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Aspects of phase-space noncommutative quantum mechanics

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ABSTRACT

In this work some issues in the context of Noncommutative Quantum Mechanics (NCQM) are addressed. The main focus is on finding whether symmetries present in Quantum Mechanics still hold in the phase-space noncommutative version. In particular, the issues related with gauge invariance of the electromagnetic field and the weak equivalence principle (WEP) in the context of the gravitational quantum well (GQW) are considered. The question of the Lorentz symmetry and the associated dispersion relation is also examined. Constraints are set on the relevant noncommutative parameters so that gauge invariance and Lorentz invariance holds. In opposition, the WEP is verified to hold in the noncommutative setup, and it is only possible to observe a violation through an anisotropy of the noncommutative parameters.

1. Introduction

Noncommutative Quantum Mechanics is an extensively studied subject [1–13] and its interest arises for many reasons, more particularly from the fact that noncommutativity is present string theory and quantum gravity and black hole models (see e.g. [14–16]). NCQM can be viewed as the low-energy and the finite number of particles limit of noncommutative field theories and its main difference from standard quantum mechanics is the inclusion of an additional set of commutation relations for position and momentum operators. The Heisenberg–Weyl algebra for these operators,

$$[\hat{x}_i, \hat{x}_j] = 0, \qquad [\hat{p}_i, \hat{p}_j] = 0, \qquad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \tag{1}$$

is deformed to the NC algebra:

$$[\hat{q}_i, \hat{q}_j] = i\theta_{ij}, \qquad [\hat{\pi}_i, \hat{\pi}_j] = i\eta_{ij}, \qquad [\hat{q}_i, \hat{\pi}_j] = i\hbar\delta_{ij}, \tag{2}$$

where θ_{ij} and η_{ij} are anti-symmetric real matrices. The two sets of variables, $\{\hat{x}_i, \hat{p}_i\}$ and $\{\hat{q}_i, \hat{\pi}_i\}$ are related by a non-canonical linear transformation usually referred to as Darboux transformation, also known as Seiberg–Witten (SW) map. It is known that, although this map is not unique, all physical observables are independent of the chosen map [11,12]. Moreover, since the NC operators are defined in the same Hilbert space as the commutative ones, one can obtain a representation of them, up to some order of the noncommutative parameters, without the need for the Darboux transformation. However, in most cases, it is simpler to use this transformation in order to recover some known aspects of quantum mechanics.

Besides the well-known operator formulation of quantum mechanics, a phase-space formulation of NCQM has been constructed [11,12] which allows for a straightforward implementation of noncommutativity. This formulation is useful for treating general problems such as, for instance, in cases where the potential is not specified. In this case, the position noncommutativity may be treated by a change in the product of functions to the Moyal $\star$-product, defined as:

$$A(x) \star_\theta B(x) := A(x) e^{(i/2)(\overleftarrow{\partial_{x_i}})\theta_{ij}(\overrightarrow{\partial_{x_j}})} B(x), \tag{3}$$

and the momentum noncommutativity is introduced via a Darboux transformation. In the case of simple potentials, the use of the Darboux transformation ensures on its own, up to some order of the noncommutative parameter, a suitable noncommutative formulation.

Throughout the following sections, whenever need, the Darboux transformation to be used is as follows [11]:

$$\hat{q}_i = \hat{x}_i - \frac{\theta_{ij}}{2\hbar}\hat{p}_j, \qquad \hat{\pi}_i = \hat{p}_i + \frac{\eta_{ij}}{2\hbar}\hat{x}_j. \tag{4}$$

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2. Gauge invariance

In order to study the effects of NCQM we shall consider some physical systems of interest and investigate the implications of the NC deformation. The first example to consider is that of a particle with mass m and charge q in a magnetic field, with the Hamiltonian given by

$$\hat{H} = \frac{1}{2m}\left[\hat{\boldsymbol{\pi}} - q\boldsymbol{A}(\boldsymbol{q})\right]^2. \tag{5}$$

In order to study this system we use the Moyal $\star$-product for the product of terms and then use the Darboux transformation, Eq. (4), to write the noncommuting Hamiltonian in terms of the commuting variables, $\hat{x}$ and $\hat{p}$. Thus, considering,

$$\hat{H}(\hat{q},\hat{\pi})\Psi(q) = \hat{H}(\hat{x},\hat{\pi}) \star_\theta \Psi(x) = \hat{H}(\hat{x},\hat{\pi})e^{(i/2)(\overleftarrow{\partial_{x_i}})\theta_{ij}(\overrightarrow{\partial_{x_j}})}\Psi(x), \tag{6}$$

at first order in the parameter θ,

$$\begin{aligned}&\left[\hat{H}(\hat{x},\hat{\pi}) + \frac{i\theta_{ab}}{2}\partial_a \hat{H}(\hat{x},\hat{\pi})\partial_b\right]\Psi(x) = \\ &= \left[\frac{1}{2m}\left(\hat{\boldsymbol{\pi}}^2 - 2q\hat{\boldsymbol{\pi}}\cdot\boldsymbol{A}(\boldsymbol{q}) + q^2A^2(q)\right)\right. \\ &\quad \left. + \frac{i\theta_{ab}}{2}\partial_a\left(q^2A^2(x) - 2q\boldsymbol{A}(x)\cdot\hat{\pi}\right)\partial_b\right]\Psi(x)\end{aligned} \tag{7}$$

If we now consider that $\theta_{ab} = \theta\epsilon_{ab}$, where ϵ_{ab} is the 2-dimensional antisymmetric symbol, the effective noncommutative Hamiltonian, at first order in θ, becomes:

$$\begin{aligned}\hat{H} &= \frac{1}{2m}\left(\hat{\boldsymbol{\pi}}^2 - 2q\hat{\boldsymbol{\pi}}\cdot\boldsymbol{A}(\boldsymbol{q}) + q^2A^2(q)\right) \\ &\quad + \frac{i}{4m}\left[\nabla\left(q^2A^2(x) - 2q\boldsymbol{A}(x)\cdot\hat{\pi}\right)\times\nabla\right]\cdot\boldsymbol{\theta}\end{aligned} \tag{8}$$

where $\boldsymbol{\theta} = \theta(1,-1,1)$. We now make use of the Darboux transformation, Eq. (4), in the momentum operator (which is now the only noncommutative operator in the Hamiltonian) to obtain:

$$\begin{aligned}\hat{H} &= \frac{1}{2m}\left[\left(\hat{\boldsymbol{p}} - q\boldsymbol{A}(\boldsymbol{x})\right)^2 - \frac{1}{\hbar}(\hat{\boldsymbol{x}}\times\hat{\boldsymbol{p}})\cdot\boldsymbol{\eta}\right. \\ &\quad \left. - \frac{q}{\hbar}(\hat{\boldsymbol{x}}\times\boldsymbol{A}(\boldsymbol{x}))\cdot\boldsymbol{\eta} + \frac{1}{4\hbar^2}\eta^2\epsilon_{ij}\epsilon_{ik}\hat{x}_j\hat{x}_k\right] \\ &\quad - \frac{1}{4m\hbar}\left[\nabla\left(q^2A^2(\boldsymbol{x}) - 2q\boldsymbol{A}(\boldsymbol{x})\cdot\hat{\boldsymbol{p}}\right.\right. \\ &\quad \left.\left. - \frac{q}{\hbar}(\hat{\boldsymbol{x}}\times\boldsymbol{A}(\boldsymbol{x}))\cdot\boldsymbol{\eta}\right)\times\hat{\boldsymbol{p}}\right]\cdot\boldsymbol{\theta},\end{aligned} \tag{9}$$

where, as in the case of θ, $\boldsymbol{\eta} = \eta(1,-1,1)$. We aim now to see how a gauge transformation modifies the Hamiltonian and study the condition under which the Hamiltonian is gauge invariant. Gauge invariance must be imposed, otherwise a gauge change would lead to a modification of the system energy for the same physical configuration. For this purpose, we consider a gauge transformation to the vector potential $\boldsymbol{A}\to\boldsymbol{A}' = \boldsymbol{A} + \nabla\alpha$, where α is a scalar function of position. Consider now the first set of terms in the Hamiltonian, Eq. (9). Under the stated transformation, we get:

$$\begin{aligned}&\frac{1}{2m}\left[\left(\hat{\boldsymbol{p}} - q\boldsymbol{A}(\boldsymbol{x}) - q\nabla\alpha\right)^2 - \frac{1}{\hbar}(\hat{\boldsymbol{x}}\times\hat{\boldsymbol{p}})\cdot\boldsymbol{\eta} -\right. \\ &\left. - \frac{q}{\hbar}(\hat{\boldsymbol{x}}\times\boldsymbol{A}(\boldsymbol{x}))\cdot\boldsymbol{\eta} - \frac{q}{\hbar}(\hat{\boldsymbol{x}}\times\nabla\alpha)\cdot\boldsymbol{\eta} + \frac{1}{4\hbar^2}\eta^2\epsilon_{ij}\epsilon_{ik}\hat{x}_j\hat{x}_k\right].\end{aligned} \tag{10}$$

If we now change the wave function on which the Hamiltonian acts, to $\Psi = e^{iq\alpha/\hbar}\Psi'$, the first set of extra terms in Eq. (9) coming from the gauge transformation will be canceled and so we may conclude that this set of therms is not problematic. However, this is not true for the second set of terms which is transformed to

$$\begin{aligned}&\left[\nabla\left(q^2(A(\boldsymbol{x}) + \nabla\alpha)^2 - 2q\boldsymbol{A}(\boldsymbol{x})\cdot\hat{\boldsymbol{p}} - 2q\nabla\alpha\cdot\hat{\boldsymbol{p}}\right.\right. \\ &\left.\left. - \frac{q}{\hbar}(\hat{\boldsymbol{x}}\times\boldsymbol{A}(\boldsymbol{x}))\cdot\boldsymbol{\eta} - \frac{q}{\hbar}(\hat{\boldsymbol{x}}\times\nabla\alpha)\cdot\boldsymbol{\eta}\right)\times\hat{\boldsymbol{p}}\right]\cdot\boldsymbol{\theta}.\end{aligned} \tag{11}$$

If we now consider the wave function transformation, $\Psi = e^{iq\alpha/\hbar}\Psi'$, we verify that the gauge transformation is not canceled due to the momentum operator outside the divergence acting on the exponential. Thus, the phase transformation that absorbs the gauge transformation terms in the first part of the Hamiltonian, Eq. (9), does not do so for the second set of terms. This comes from the fact that, in the first term, the change in $\boldsymbol{A}$ can be seen as a change in $\hat{\boldsymbol{p}}$, and a constant change in momenta can always be absorbed by a phase change. The same does not occur for the change in the second term, making it impossible to accommodate it into a change in phase. Therefore, in order to make the Hamiltonian gauge invariant, this term must vanish. To accomplish this for any $\boldsymbol{A}$, θ must vanish. This result is consistent to an explicit computation in the context of the Hamiltonian of fermionic fields [17].

3. Gravitational quantum well and the equivalence principle in NCQM

A very interesting system to directly connect gravity to quantum mechanics is the gravitational quantum well [18–20]. As we shall see, this connection can be used to constrain quantum measurements of gravity phenomena and to test the equivalence principle (see also Refs. [21,22]). It is easy to show that this principle holds for usual quantum mechanics, in the sense that a gravitational field is equivalent to an accelerated reference frame. We shall see that this also holds in the context of NCQM for isotropic noncommutativity parameters. In the following we shall study the noncommutative GQW [10] and its connection to accelerated frames of reference.

3.1. Fock space formulation of NC gravitational quantum well

Let us consider the GQW in the context of NCQM. To start with we review some aspects of the usual GQW in standard quantum mechanics. The Hamiltonian is given by:

$$\hat{H} = \frac{1}{2m}\hat{\boldsymbol{p}}^2 + mg\hat{x}_i \tag{12}$$

for a particle with mass, m, in a gravitational field with acceleration, g, in the x_i direction.

With the Fock space treatment in mind we define creation and annihilation operators for this Hamiltonian:

$$\hat{b} = \left(\frac{m^2}{\hbar^2 g}\right)^{\frac{1}{3}}\left[\hat{x} + \frac{i}{2}\left(\frac{g^2\hbar}{m^4}\right)^{\frac{1}{3}}\hat{p}_x\right], \tag{13}$$

$$\hat{b}^\dagger = \left(\frac{m^2}{\hbar^2 g}\right)^{\frac{1}{3}}\left[\hat{x} - \frac{i}{2}\left(\frac{g^2\hbar}{m^4}\right)^{\frac{1}{3}}\hat{p}_x\right], \tag{14}$$

where the definition concerns only for the x direction, as the y component of the Hamiltonian is just that of a free particle.