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Uniform gradient expansions

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ABSTRACT

Cosmological singularities are often discussed by means of a gradient expansion that can also describe, during a quasi-de Sitter phase, the progressive suppression of curvature inhomogeneities. While the inflationary event horizon is being formed the two mentioned regimes coexist and a uniform expansion can be conceived and applied to the evolution of spatial gradients across the protoinflationary boundary. It is argued that conventional arguments addressing the preinflationary initial conditions are necessary but generally not sufficient to guarantee a homogeneous onset of the conventional inflationary stage.

The dynamical approach to the cosmological singularity has been historically investigated in terms of an expansion in spatial gradients of the geometry [1,2] (see also [3]). Denoting with t the cosmic time coordinate, the gradient expansion in the proximity of the big-bang singularity is formally defined in the limit $t \to 0$ where the spatial gradients turn out to be subdominant in comparison with the extrinsic curvature. This important observation implies that close to the singularity the geometry may be highly anisotropic but rather homogeneous [1,2]. As soon as an inflationary event horizon is formed, the physical rationale for a complementary gradient expansion emerges in the limit $t \to \infty$ [4–6]. This idea is applied, for instance, when arguing in favour of the so-called cosmic no-hair conjecture stipulating that in conventional inflationary models any finite portion of the event horizon gradually loses the memory of an initially imposed anisotropy or inhomogeneity so that the metric attains the observed regularity regardless of the initial boundary conditions (see Ref. [7] for this formulation of the conjecture and also Refs. [8,9] for some other early contributions). According to the standard lore, one of the central motivations of the whole inflationary paradigm (see e.g. [10, 11]) is to wash out primeval anisotropies in the expansion right after the formation of the inflationary event horizon (see, however, Ref. [12] for a critical perspective on the limitations of the no-hair conjecture).

Over a time scale $\mathcal{O}(t_*)$ corresponding to the formation of the inflationary event horizon, it is therefore plausible to analyze the space–time geometry not only in terms of a backward gradient expansion (valid for $t < t_*$) and but also by means of a forward expansion (applicable for $t > t_*$). In both regimes, following the synchronous Adler–Deser–Misner parametrization [13] the four-dimensional metric tensor can be decomposed as $g_{00} = 1$, $g_{ij} = -\gamma_{ij}(\vec{x}, t)$ and $g_{0i} = 0$. The six independent entries of $\gamma_{ij}(\vec{x}, t)$ can be expanded as:

$$\gamma_{ij}(\vec{x}, t) = a^2(t)\left[\alpha_{ij}(\vec{x}) + \beta_{ij}(\vec{x}, t) + \dots\right], \tag{1}$$

where $\beta_{ij}(\vec{x}, t)$ contains two spatial gradients and the ellipses stand for the higher terms in the expansion containing a progressively larger (even) number of spatial gradients. Once the inhomogeneous seed metric $\alpha_{ij}(\vec{x})$ is assigned, the Einstein equations together with the equations of the sources determine $\beta_{ij}(\vec{x}, t)$ whose explicit form can always be parametrized in terms of two dimensionless functions that shall be conventionally called $f(t)$ and $g(t)$:

$$\beta_i^j(\vec{x}, t) = f(t)\frac{\mathcal{P}_i^j(\vec{x})}{H_*^2} + g(t)\frac{\mathcal{P}(\vec{x})}{H_*^2}\delta_i^j, \tag{2}$$

where $\mathcal{P}_i^j(\vec{x}) = a^2(t)\mathcal{R}_i^j(\vec{x}, t)$ is expressed in units of $H_*^2 \simeq t_*^{-2}$ and $\mathcal{R}_{ij}(\vec{x}, t)$ denotes the three-dimensional Ricci tensor. The evolution of $f(t)$ and $g(t)$ depends, in its turn, on the zeroth-order solution. If the expansion of Eqs. (1) and (2) can be safely applied, the Universe is already quasi-homogeneous in a time interval centred around t_* and this will be our first assumption on the process describing the formation of the event horizon. Secondly we shall posit that, for $t < t_*$, the zeroth-order solution expands in a decelerated manner while it inflates for $t > t_*$: roughly speaking this assumption implies that t_* can be identified with the protoinflationary boundary. We shall finally admit that the zeroth-order

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evolution of the extrinsic curvature is continuous and monotonic: this last assumption can be relaxed but it is nonetheless realized in the explicit toy examples illustrated hereunder. Are the three aforementioned assumptions sufficient to guarantee that the spatial gradients of the geometry are exponentially suppressed for $t \gg t_*$? Are they compatible with the asymptotic suppression of the spatial gradients during the quasi-de Sitter stage? The two previous questions can be approached within a uniform gradient expansion holding across the protoinflationary boundary.

While the fully inhomogeneous inflationary initial conditions represent a rather complicated topic whose proper formulation is beyond the scopes of this paper, in what follows we shall content ourselves with the conventional lore in a system where the inflaton field φ evolves under the action of its own potential $W(\varphi)$ and in the presence of spatial inhomogeneities (characterized by the three-dimensional Ricci scalar $\mathcal{R}$); to account for a possible decelerated behaviour in the preinflationary epoch, we shall also include the contribution of an ambient relativistic fluid whose energy density will be denoted by ρ. When the various components of the system are all in equipartition we approximately have[1]:

$$\dot{\varphi}^2 \simeq W(\varphi) \simeq \rho \simeq \mathcal{R}\overline{M}_{\mathrm{P}}^2, \tag{3}$$

where the overdot denotes a derivation with respect to the cosmic time coordinate t. Since the kinetic energy, the spatial curvature and the fluid energy density are all diluted faster than $W(\varphi)$, Eq. (3) implies, in the conventional lore, that the Universe inflates before becoming inhomogeneous; this conclusion holds provided the background was expanding prior to t_*. A successful inflationary dynamics can also be realized in other situations compatible with Eq. (3) like, for instance, $W(\varphi) \gg \dot{\varphi}^2 \simeq \rho \simeq \mathcal{R}\overline{M}_{\mathrm{P}}^2$: also in this case all the components of the energy–momentum tensor will quickly disappear and the potential will dominate even faster than in the case of Eq. (3). Conversely, if the approximate equipartition of Eq. (3) is violated, the typical scale of the potential gets much smaller than the other components of the system: various inverted hierarchies can be envisaged and they turn out to be particularly relevant in the case of plateau-like potentials [11]. For instance it can happen that $\dot{\varphi}^2 \simeq \rho \simeq \mathcal{R}\overline{M}_{\mathrm{P}}^2 \gg W$: in this case the kinetic energy is diluted more rapidly than the other terms and, after few efolds, the spatial gradients contained in $\mathcal{R}$ dominate the evolution of the sources while the potential is still too small to play any role so that the Universe fails to inflate.[2] For the present ends what matters is not the likelihood of inflation (or its naturalness) given a generic set of initial data but just the observation that Eq. (3) and its descendants are based on the scaling properties of the various components of the total energy–momentum tensor under the implicit assumption that the geometry is already expanding. We shall therefore grant that the initial stages of the inflationary phase are continuously preceded by an epoch where the geometry expands in a decelerated manner and study, in this standard framework, the evolution of the spatial gradients.

Within the conventional formulation of the inflationary initial conditions it can be naively expected that $f(t)$ and $g(t)$ will be going to zero as a power (for $t < t_*$) and quasi-exponentially (for $t > t_*$). The governing equations of the system imply that the evolution of $g(t)$ depends directly on the sources (see below, Eqs. (22)–(24)) while in the case of $f(t)$ the evolution reads:

$$\ddot{f} + 3H\dot{f} + 2H_*^2\left(\frac{a_*}{a}\right)^2 = 0, \qquad H = \frac{\dot{a}}{a}. \tag{4}$$

Introducing the initial integration time t_i, the solution of Eq. (4) depends on $f_i = f(t_i)$ and $\dot{f}_i = \dot{f}(t_i)$ and can be written as:

$$\dot{f}(t) = \dot{f}_i\left(\frac{a_i}{a}\right)^3 - 2H_*^2\left(\frac{a_*}{a}\right)^3 \int_{t_i}^{t} \frac{a(t')}{a_*}\,dt',$$

$$f(t) = f_i + a_*^3\dot{f}_i \int_{t_i}^{t} \frac{dt'}{a^3(t')} - 2H_*^2 a_*^2 \int_{t_i}^{t} \frac{dt'}{a^3(t')} \int_{t_i}^{t'} a(t'')\,dt''. \tag{5}$$

The explicit form of $a(t)$ is obtainable by solving the zeroth-order in the gradient expansion but let us just assume that $\ddot{a} < 0$ and $\dot{a} > 0$ for $t < t_*$. Such a functional behaviour is realized, for instance,[3] when $a(t) \sim a_*(t/t_*)^{1/\delta}$ provided $1 < \delta \leq 3$. For $t > t_*$ we posit instead that $\ddot{a} > 0$ and $\dot{a} > 0$ and the conventional inflationary dynamics implies $\epsilon = -\dot{H}/H^2 \lesssim 1$. Under the conditions expressed by Eq. (3) the solution of Eq. (4) in the two asymptotic limits, naively implies[4]:

$$\lim_{a \ll a_*} f(a) \to \left(\frac{a}{a_*}\right)^{2(\delta-1)}, \qquad \lim_{a \gg a_*} f(a) \to \left(\frac{a}{a_*}\right)^{-2+2\epsilon}. \tag{6}$$

Not surprisingly, Eq. (6) is consistent with the results separately obtainable in the two limits (see, e.g. [1,2] and [4–6]) but what matters here is that such a condition seems to demand the existence of an extremum for $a \sim \mathcal{O}(a_*)$ or $t \simeq \mathcal{O}(t_*)$. According to Eq. (5) the existence of a maximum would imply that $|\dot{f}(t)| \to 0$ for $t \simeq t_*$, where the absolute value accounts for the possibility of negative values of $f(t)$. The vanishing of $\dot{f}(t)$ can occur either for finite cosmic time (but then we must have that $\dot{f}_i \neq 0$) or asymptotically for $t \gg t_*$. The choice $\dot{f}_i \neq 0$ causes the presence of divergent term in the limit $t \ll t_*$ and this clashes with the possibility of imposing quasi-homogeneous initial conditions in the preinflationary phase, as conventionally assumed. According to this argument, what can happen, at most is $|\dot{f}| \to 0$ for $t \gg t_*$; if this is the case the gradients will not be asymptotically suppressed but $f(t)$ will rather reach a constant value. Thus the smooth and monotonic evolution of the extrinsic curvature across the protoinflationary transition does not seem sufficient to guarantee that the spatial gradients will be exponentially suppressed during the fully developed inflationary phase. The simplistic way of reasoning pursued in this paragraph assumes, without proof, a certain behaviour of the scale factor. In what follows we shall then focus the attention to the full zeroth-order and first-order solutions in the case when the extrinsic curvature interpolates between a decelerated regime and an accelerated evolution in the vicinity of t_*.

We are now ready to consider the general system of equations: separating the extrinsic curvature ($K_{ij} = -\dot{\gamma}_{ij}/2$) from the contribution of the intrinsic curvature ($\mathcal{R}_{ij}$), the (00) and (0i) components of the contracted Einstein equations read:

$$\dot{K} - \mathrm{Tr}K^2 = \ell_{\mathrm{P}}^2\left[\frac{(3p+\rho)}{2} + (p+\rho)u^2 + \dot{\varphi}^2 - W(\varphi)\right], \tag{7}$$

[1] The Planck mass will be defined as $\overline{M}_{\mathrm{P}} = 1/\sqrt{8\pi G}$ where G is the Newton constant; the Planck length, in these natural units, is just the inverse of $\overline{M}_{\mathrm{P}}$, i.e. $\ell_{\mathrm{P}} = \sqrt{8\pi G}$.

[2] Other potentially dangerous hierarchies are, for instance, $\dot{\varphi}^2 \gg W \gg \mathcal{R}\overline{M}_{\mathrm{P}}^2 \simeq \rho$ and $\dot{\varphi}^2 \gg W \simeq \mathcal{R}\overline{M}_{\mathrm{P}}^2 \gg \rho$.

[3] Note, incidentally, that if the preinflationary background is dominated by a perfect fluid with constant barotropic index w, then $\delta = 3(w+1)/2$; conversely if the preinflationary background is dominated by the kinetic energy of the inflaton (and the ambient fluid is absent) $\delta \to 3$.

[4] If regarded in cosmic time, the requirements of Eq. (6) translate in an approximate interpolating form of $f(t)$ that could be written, up to slow roll corrections, as $f(t) \simeq (t/t_*)^{2(\delta-1)/\delta+1}/[e^{2H_* t} - 1]$. As we shall demonstrate, this plausible guess, implying $\dot{f}(t) \simeq 0$ for $t \simeq t_*$, is not supported by the explicit dynamics of the spatial gradients.