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A nonperturbative method for the scalar field theory

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ABSTRACT

We compute an all order correction to the scalar mass in the Φ^4 theory using a new method of functional integration adjusted also to the large couplings regime.

1. Introduction

Currently very much is known about the perturbative behavior of many theories with or without gauge fields. Beta functions for the Φ^4 theory and QED are known up to the fifth order whereas for QCD, up to the fourth order [1–7]. However, there is a limited knowledge regarding the nonperturbative behavior of the same theories. Recently attempts [8] have been made for determining the existence in some renormalization scheme of all order beta functions for gauge theories with various representations of fermions. It is rather useful to search for alternative methods, which may reveal either the higher orders of perturbation theories or even the nonperturbative regime.

Here we shall consider the massive Φ^4 theory as a laboratory for implementing a method that can be further applied to more comprehensive models. There is an ongoing debate with regard to the behavior of the renormalized coupling λ at small momenta referred to as "the triviality problem" [9–11]. With the hope that our approach might shed light even on this problem, we introduce a new variable in the path integral formalism which allows for a more tractable functional integration and series expansion. Then we compute in this new method the corrections to the mass of the scalar in all order of perturbation theory. This approach should be regarded as an alternate renormalization procedure. Since the corresponding mass anomalous dimension $\gamma(m^2) = \frac{d \ln m^2}{d\mu^2}$ has the first order (one loop) universal coefficient, we verify that the first order correction is correct. However, we expect that the next orders are different.

2. The set-up

We shall illustrate our approach for a simple scalar theory, given by the Lagrangian:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1, \qquad \mathcal{L}_0 = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2} m_0^2 \Phi^2, \qquad \mathcal{L}_1 = -\frac{\lambda}{4!}\Phi^4. \tag{1}$$

For convenience, we will work both in the Minkowski and Euclidian space.

The generating functional in the Euclidian spaces has the expression:

$$W[J] = \int d\Phi \exp\left[-\int d^4x \left[\frac{1}{2}\left(\frac{\partial \Phi}{\partial \tau}\right)^2 + \frac{1}{2}(\Delta \Phi)^2 + \frac{1}{2} m_0^2 \Phi^2 + \frac{\lambda}{4!}\Phi^4 + J\Phi\right]\right] \tag{2}$$

and can be written as

$$W[J] = \exp\left[\int d^4x \mathcal{L}_1\left(\frac{\delta}{\delta J}\right)\right] W_0[J] \tag{3}$$

where

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$$W_0[J] = \int d\Phi \exp\left[\int d^4x(\mathcal{L}_0 + J\Phi)\right]. \tag{4}$$

From Eq. (3) is clear how the perturbative approach can work. If λ is a small parameter, one can expand the exponential in terms of λ and solve successive contributions accordingly. However, we are interested in the regime where λ is large and one cannot use the above expansion.

We will illustrate our approach simply on a simple function. Assume that we have the following one-dimensional integral which cannot be solved analytically:

$$I = \int dx \exp\left[-af(x)\right], \tag{5}$$

where f is polynomial of x. For a small, the expansion in a makes sense. For $a \to \infty$, the Taylor expansion uses:

$$\lim_{a\to\infty} \frac{d^n \exp[-af(x)]}{da^n} = 0 \tag{6}$$

which does not lead to a correct answer.

We shall use, however, a simple trick. We replace in the polynomial f some of the variables x with a new variable y (for example $x^4 \to x^2y^2$). Then we write:

$$I = \int dxdy\delta(x-y)\exp\left[-af(x,y)\right] = \int dxdydz \exp\left[-i(x-y)z\right]\exp\left[-af(x,y)\right] = \int dxdydz \exp\left[-i(x-y)z - af(x,y)\right] \tag{7}$$

This does not help too much in the present form. However, if $f(x,y) = x^2y^2$ or any other function that contains x^2, we can form the perfect square:

$$-ixz - ax^2y^2 = -\left(\sqrt{a}xy + \frac{iz}{2\sqrt{a}y}\right)^2 - \frac{z^2}{4ay^2}. \tag{8}$$

Introduced in Eq. (7) this leads:

$$I = \text{const}\int d\frac{1}{\sqrt{a}y}dz \exp\left[-\frac{z^2}{4ay^2}\right]\exp[iyz] \tag{9}$$

Then expansion in $\frac{1}{a}$ makes sense and one can write:

$$I = \text{const}\int dxdz\frac{1}{\sqrt{a}y}\left[1 - \frac{z^2}{4ay^2} + \ldots\right]\exp[iyz] \tag{10}$$

This expansion may seem ill defined and highly divergent. For example, if one integrates over z, then one already encounters infinities. However, in the functional method one is dealing with functions instead of simple variables and one encounters divergences also in the usual expansion in small parameters. We will consider the above approach as our starting point and solve the problem of divergences as they appear.

We will start with the simple partition function for a Φ^4 theory without a source:

$$W[0] = \int d\Phi \exp\left[i\int d^4x[\mathcal{L}_0 + \mathcal{L}_1]\right] \tag{11}$$

We consider the extended functional δ defined in the Minkowski space as (see Appendices A and B):

$$\delta(\Phi) = \text{const}\int dK \exp\left[i\int d^4x_M K\Phi\right] \tag{12}$$

which in the Euclidian space becomes:

$$\delta(\Phi) = \text{const}\int dK \exp\left[-\int d^4xK\Phi\right] \tag{13}$$

We then rewrite Eq. (11) in Minkowski space as

$$\begin{aligned} W[0] &= \int d\Phi d\Psi\delta(\Phi-\Psi)\exp\left[i\int d^4x\left[\mathcal{L}_0 - \frac{\lambda}{8}\Phi^2\Psi^2\right]\right] \\ &= \text{const}\int d\Phi d\Psi dK \exp\left[i\int d^4xK(\Phi-\Psi)\right]\exp\left[i\int d^4x\left[\mathcal{L}_0 - \frac{\lambda}{8}\Phi^2\Psi^2\right]\right] \\ &= \text{const}\int \frac{1}{\sqrt{\lambda}}d\Phi dK \exp\left[i\int d^4x\frac{2}{\lambda}K^2\right]\exp\left[i\int d^4xK\Phi^2\right]\exp\left[i\int d^4x\mathcal{L}_0\right] \end{aligned} \tag{14}$$

In order to obtain this result, we made the following change of variable in the second line of Eq. (14): $K \to K\Phi$, $\Psi \to \frac{\Psi}{\Phi\sqrt{\lambda}}$. Note that the λ term gets rescaled by 3 such that to take into account the various contribution of the Fourier modes.

We will estimate the first order of the integral in Eq. (14) given by: