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Proton–proton fusion in lattice effective field theory

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ABSTRACT

The proton–proton fusion rate is calculated at low energy in a lattice effective field theory (EFT) formulation. The strong and the Coulomb interactions are treated non-perturbatively at leading order in the EFT. The lattice results are shown to accurately describe the low energy cross section within the validity of the theory at energies relevant to solar physics. In prior works in the literature, Coulomb effects were generally not included in non-perturbative lattice calculations. Work presented here is of general interest in nuclear lattice EFT calculations that involve Coulomb effects at low energy. It complements recent developments of the adiabatic projection method for lattice calculations of nuclear reactions.

1. Introduction

Calculations of nuclear reactions from a microscopic theory are of fundamental importance. Nuclear cross sections are important in understanding the observed abundances of elements [1–5]. These reactions occur under conditions of extreme densities and temperatures where all the known fundamental forces of nature – gravitation, electro-weak interactions, and strong interactions – play a role. Thus nuclear reaction cross sections impact disparate areas of physics such as astrophysics, nuclear physics and particle physics in a crucial manner. The effective field theory (EFT) formulation of the microscopic nuclear interaction plays a central role in the nuclear reaction calculations [6–11]. EFT provides a model-independent framework where one can make reliable estimates of the theoretical error. This is important as many of the nuclear reactions occur under extreme conditions that cannot be reproduced in terrestrial laboratories. Nuclear astrophysical models require reliable handle on the nuclear theory errors [2,4,12]. Further, EFT provides a bridge between nuclear physics and particle physics where nuclear observables can be connected to particle physics parameters such as the quark masses [13].

Applications of EFT in the few-nucleon systems have been quite successful [6–11]. Though there is a good understanding of the microscopic nuclear interactions, their application to larger nuclear systems poses serious computational challenges. Numerical lattice methods from particle physics combined with EFT provide a promising possibility. The lattice EFT formulation allow a systematic error analysis derived from EFT. Ground and excited state energies for several atomic nuclei have been calculated accurately [14–16]. Many-body properties in dilute neutron matter have also been addressed [17]. Recently progress has been made in calculating nuclear reactions using lattice methods albeit in simple systems [18–20]. The proposal in Refs. [18,19] is to first construct an effective two-body Hamiltonian from first principle using an adiabatic projection method. This Hamiltonian is then used to calculate elastic and inelastic reactions involving nuclei such as $a+b \to \gamma + c$, $a+b \to c+d$ with a, b, c and d being atomic nuclei, and γ a photon. In this work we consider the contribution from the long range Coulomb force. Nuclear reactions involving compound nuclei will necessarily involve Coulomb interactions that become non-perturbative at energies relevant to astrophysics. To test the basic formulation we calculate proton–proton elastic scattering and fusion at low energy. This simpler system allows us to isolate the Coulomb effect without a complicated nuclear strong force.

The pioneering calculation by Bethe and Critchfield showed that proton–proton fusion $p+p \to d + e^+ + \nu_e$ powers the sun [21,22]. This is a rare weak process that is the first crucial step in solar fusion. A small Coulomb barrier along with the slow rate of the weak process leads to a long and steady burning of hydrogen in medium mass stars such as our sun [23]. The proton fusion rate is crucial to understanding solar neutrino production and its subsequent detection in terrestrial laboratories [24].

Bahcall and May refined the fusion rate calculation [25] and set the benchmark for future evaluations such as Refs. [12,26]. The capture rate was expressed in terms of model-independent

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parameters such as the deuteron binding momentum, the proton–proton scattering length, etc., that are not sensitive to the details of the nuclear potential. The proton–proton fusion rate was analyzed in EFT with short-ranged interactions in Refs. [27,28]. The EFT calculations matched the work by Bahcall and May when expressed in terms of the two-body scattering parameters and one-body currents. Higher order corrections from two-body currents have also been included in EFT calculations in a systematic manner, see Refs. [29–31]. We consider the leading order (LO) contribution in lattice EFT. Both the strong and Coulomb interaction are non-perturbative at LO. The higher order contributions are perturbative [27–31] and should not pose any technical challenge in future lattice calculations.

2. Interaction

Proton–proton fusion at solar energies around the Gamow peak is dominated by capture from the s-wave. At these energies $E \sim 6$ keV, the initial state proton–proton strong interaction at LO is described by the Lagrangian [32,33]:

$$\mathcal{L} = \psi^\dagger\left[i\partial_0 + \frac{\nabla^2}{2M}\right]\psi - \frac{c_0}{4}(\psi\sigma_2\psi)^\dagger(\psi\sigma_2\psi), \tag{1}$$

where the proton mass $M = 938.3$ MeV, and ψ represents the spin-1/2 protons. The Pauli matrix σ_2 is used to project the protons onto the spin-singlet channel. We use natural units with $\hbar = 1 = c$. The strong interaction potential in coordinate space for proton–proton scattering in the s-wave spin-singlet channel, corresponding to Eq. (1), is

$$V_s(\vec{\boldsymbol{r}}) = c_0\delta(\vec{\boldsymbol{r}}). \tag{2}$$

The long range Coulomb force is described by the Coulomb potential

$$V_c(\vec{\boldsymbol{r}}) = \frac{\alpha}{r}, \tag{3}$$

with $\alpha = 1/137$. The coupling c_0 can be determined in the continuum from proton–proton scattering length a_p [32,33]. Given these interactions, we construct the lattice theory by discretizing space in a periodic box.

The kinetic energy term in the Hamiltonian is written as a Laplacian constructed out of forward–backward differences on the lattice:

$$\begin{aligned}&-\frac{1}{2M}\int d^3r\, \psi(\vec{\boldsymbol{r}})^\dagger\nabla^2\psi(\vec{\boldsymbol{r}})\\ &\quad\to -\frac{1}{2\hat{M}b}\sum_{\vec{\boldsymbol{n}},\hat{\boldsymbol{l}}}\hat{\psi}^\dagger(\vec{\boldsymbol{n}})\big[\hat{\psi}(\vec{\boldsymbol{n}}+\hat{\boldsymbol{l}})+\hat{\psi}(\vec{\boldsymbol{n}}-\hat{\boldsymbol{l}})-2\hat{\psi}(\vec{\boldsymbol{n}})\big],\end{aligned} \tag{4}$$

where b is the lattice spacing, the integer vector $\vec{\boldsymbol{n}}$ indicates the lattice sites, $\hat{\boldsymbol{l}}$ is a unit vector in the x-, y-, z-direction. The hatted quantities $\hat{M}$, $\hat{\psi}$ are in lattice units. The review article in Ref. [34] has more details. The strong interaction potential reduces to a Kronecker delta function at the origin on the lattice. The Coulomb potential is defined on the discretized lattice in a straightforward manner. However, at the origin we regulate it, i.e., replace it by a Kronecker delta function with a coupling d_0 to be determined later. In the presence of both strong and Coulomb potentials, only the linear combination of $c_0 + d_0$ determines phase shifts and amplitudes. This is a consequence of the overlap of the ultraviolet divergences in the strong and Coulomb interactions in the EFT [32, 33]. The lattice Hamiltonian is defined on a periodic box of size L in lattice units.

Proton–proton fusion involves a deuteron in the final state that can be described in the EFT accurately [35]. The LO spin-triplet interaction can be described with a short-ranged interaction $(\psi\sigma_2\sigma_i\phi)^\dagger(\psi\sigma_2\sigma_i\phi)$ where ϕ is the spin-1/2 neutron field. The coupling for this spin-triplet interaction is tuned independently of the spin-singlet interaction in Eq. (1) to reproduce the deuteron binding energy $B = 2.2246$ MeV [36]. The deuteron bound state can be described in the lattice formulation of the short-ranged interaction as well.

3. Scattering and fusion

Elastic scattering is commonly described in lattice calculations using Lüscher's method [37,38]. The energy shifts in a periodic box in the presence of a short-ranged interaction is used to calculate the elastic phase shifts. Perturbative Coulomb contributions to two-particle scattering in a finite volume have been considered recently [39] but a general method for calculating Coulomb interactions non-perturbatively at low energy using Lüscher's method doesn't exist. Here we calculate the non-relativistic phase shift in the presence of the long range Coulomb force using a hard spherical wall boundary condition. This method was used in Refs. [14, 40] to calculate two-body phase shift due to the short-ranged strong interaction. The spherical wall method was found to be better suited than Lüscher's method for problems involving coupled channels in Refs. [14] in lattice calculation.

To understand the spherical wall method, consider a short-ranged potential $V_s(r)$ inside a hard spherical wall of radius R [14, 40]. The continuum asymptotic s-wave solution to the Schrödinger equation inside the hard wall has the form

$$\left|\psi_s(\vec{\boldsymbol{r}})\right| = \left|j_0(kr)\cos\delta_s - y_0(kr)\sin\delta_s\right|, \tag{5}$$

in terms of the spherical Bessel functions j_0 and y_0 at the center-of-mass momentum k. At the spherical wall boundary R, the wave function must vanish, giving

$$\tan\delta_s(k) = \frac{j_0(kR)}{y_0(kR)}. \tag{6}$$

On a cubic lattice one cannot fit a sphere of radius R. Instead for a given spherical hard wall of radius R, following Ref. [14], one defines an adjustable wall radius R_w where the free wave function vanishes:

$$j_0(k_0R_w) = 0 \quad \Rightarrow \quad R_w = \frac{\pi}{k_0}. \tag{7}$$

k_0 is the center-of-mass momentum of the free theory on the lattice. It corresponds to the momentum of the first energy of the spectrum on the lattice. The self-consistent use of R_w in Eq. (6) by setting $R = R_w$ is shown to accurately reproduce the strong interaction phase shifts for various two-nucleon channels [14]. We follow the same procedure in calculating the strong-Coulomb phase shift for proton–proton scattering on the lattice with a box size greater than the diameter of the spherical shell.

Traditionally, proton–proton scattering is described by considering the Coulomb subtracted phase shift $\delta_{sc} = \delta_{\text{full}} - \delta_c$. The full phase shift specifies the scattering amplitude $\mathcal{T}$ through the relation

$$\mathcal{T}(k) = \frac{2\pi}{\mu}\frac{\exp(i2\delta_{\text{full}}) - 1}{2ik}, \tag{8}$$

where $\mu = M/2$ is the reduced mass. The purely Coulomb phase shift $\delta_c(k) = \text{Arg}[\Gamma(1 + i\eta_k)]$ is independent of the short-ranged