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Massive renormalization scheme and perturbation theory at finite temperature

Jean-Paul Blaizot [a,*], Nicolás Wschebor [b]

[a] Institut de Physique Théorique, CNRS/URA2306, CEA-Saclay, 91191 Gif-sur-Yvette, France
[b] Instituto de Fisica, Faculdad de Ingeniería, Universidade de la República, 11000 Montevideo, Uruguay

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ABSTRACT

We argue that the choice of an appropriate, massive, renormalization scheme can greatly improve the apparent convergence of perturbation theory at finite temperature. This is illustrated by the calculation of the pressure of a scalar field theory with quartic interactions, at 2-loop order. The result, almost identical to that obtained with more sophisticated resummation techniques, shows a remarkable stability as the coupling constant grows, in sharp contrast with standard perturbation theory.

1. Introduction

Over the last two decades considerable efforts have been made to understand the behavior of perturbation theory in quantum field theory at finite temperature (for reviews, see [1–3]). These efforts are in part motivated by the physics of the quark–gluon plasma, and the hope that QCD asymptotic freedom would allow for reliable calculations at sufficiently high temperature. It is well known, however, that in QCD infrared divergences inevitably occur and eventually cause a breakdown of perturbation theory at a finite order in the expansion in powers of the coupling constant [4]. But even in theories where such a breakdown does not occur, such as in scalar theories, perturbation theory at finite temperature appears as poorly convergent as in QCD.

Two routes have been followed to try to overcome these difficulties. The first one is to include more and more terms into the perturbative series, hoping in doing so to compensate for the poor apparent convergence. Thus, the pressure of the massless scalar theory with a quartic interaction is now known up to order $g^8 \log(1/g)$ (see Ref. [5] and references therein). While definite improvements are observed at small coupling when high orders are taken into account, the bad behavior of the perturbative series resurfaces as soon as the coupling gets moderately large.

The other route involves various reorganizations of the perturbative expansion, such as screened perturbation theory [6], infinite resummations, the use of functional variational techniques such as the so-called 2PI (two particle irreducible) formalism [7], or the non-perturbative renormalization group (NPRG) [8]. Remarkably, all these approaches produce results that remain stable as one increases the coupling constant, in sharp contrast with strict perturbation theory. At the same time, some of these calculations suggest that the physics at moderate coupling is minimally non-perturbative. In particular, calculations using the functional renormalization group within the most sophisticated approximation scheme available [8,9] yield results that do not deviate much from simple self-consistent quasiparticle approximations (such as the lowest order 2PI approximation, or the local potential approximation of the NPRG [10]).

These latter results suggest to look for an underlying simplicity, and it is indeed the purpose of this paper to report on progress in this direction. We shall argue that the difficulties encountered in finite temperature calculations can be attributed to a large extent to inappropriate choices of renormalization schemes. The success of the NPRG invites us to look for a scheme where the thermal mass plays a central role, and also where a decoupling of modes occurs when the typical scales exceed the temperature, both features that are automatically included in the NPRG. We shall exhibit such a massive renormalization scheme and show that it yields indeed a well behaved perturbative expansion. This will be illustrated in this letter with the calculation of the pressure of a scalar field theory. More elaborate calculations will be presented in forthcoming publications.

* Corresponding author.
E-mail address: jean-paul.blaizot@cea.fr (J.-P. Blaizot).

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2. Massive thermal scheme

Let us start by recalling the origin of the difficulties with standard perturbation theory at finite temperature, focusing on a scalar theory of massive modes with quartic interactions $\sim g^2\varphi^4$ (see Eq. (4) below). The expansion parameter, which is not simply g^2, depends on the magnitude of the average fluctuations of the field, given by (we ignore here the vacuum fluctuations)

$$\langle\varphi^2\rangle_\kappa \approx \int^{\kappa} \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{n_\mathbf{p}}{E_\mathbf{p}} \approx T\kappa, \qquad n_\mathbf{p} = \frac{1}{e^{E_\mathbf{p}/T}-1}, \tag{1}$$

where $E_\mathbf{p} = \sqrt{m^2+\mathbf{p}^2}$, with m the mass, κ is an ultraviolet momentum cutoff and the approximation for $\langle\varphi^2\rangle_\kappa$ is valid for $\kappa \lesssim T$. Using this, we can define a dimensionless expansion parameter, γ_κ, as the ratio between the interaction energy ($\sim g^2\langle\varphi^2\rangle_\kappa^2$), and the kinetic energy ($\sim\kappa^2\langle\varphi^2\rangle_\kappa$) of modes with typical momentum κ,

$$\gamma_\kappa \sim \frac{g^2\langle\varphi^2\rangle_\kappa}{\kappa^2} \sim \frac{g^2 T}{\kappa}. \tag{2}$$

Thus for $\kappa \sim T$, the expansion parameter is essentially the coupling constant $\gamma_T \sim g^2$. However, γ_κ grows as κ decreases. Eventually γ_κ becomes of order unity when $\kappa \sim g^2 T$, at which point standard perturbation theory breaks down.

However, at least in the case of scalar theories, this scenario is too pessimistic, and this for two reasons. Observe first that when $m \ll \kappa \lesssim T$ the theory behaves as a massless three-dimensional theory with (dimensionful) coupling $g^2 T$. The associated *dimensionless* coupling can be identified to γ_κ, and it obeys the one-loop renormalization group equation (see Eq. (23) below)

$$\kappa\frac{d\gamma_\kappa}{d\kappa} = -\gamma_\kappa + \frac{3}{16}\gamma_\kappa^2. \tag{3}$$

The first term in this equation results from the analysis that we just presented, the second term is the one loop correction. This correction tames the growth of the coupling suggested by the first term, and indeed the infrared fixed-point at $\gamma_* = 16/3$ prevents the blow-up of γ_κ. The success of the expansion in $\epsilon = 4-d$ indicates that perturbation theory in the vicinity of this fixed point is reasonably accurate [11]. The second reason which prevents the breakdown of perturbation theory is of course the generation of a thermal mass m of order gT which freezes the running of the coupling at the scale $\kappa \sim m$.[1]

These considerations concerning the mechanisms that prevent the growth of the coupling, make paradoxical the fact that standard perturbation theory behaves so badly at finite temperature. In fact, as we have already alluded to, the reason may not be perturbation theory itself, but rather the particular scheme used. Most studies are done in non-decoupling schemes, such as the $\overline{\text{MS}}$ scheme, which is popular because of its technical simplicity. But the discussion above suggests the use of a scheme where the matching between the four-dimensional and the three-dimensional regimes when $\kappa \lesssim T$, as well as the suppression of fluctuations when $\kappa \lesssim m$, are manifest order by order in perturbation theory. We shall now present such a scheme.

We consider the theory of a scalar field φ with the action

$$S[\varphi] = \int_0^\beta d\tau \int d^dx \left\{\frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi + \frac{m_B{}^2}{2}\varphi^2 + \frac{g_B^2}{4!}\varphi^4\right\}, \tag{4}$$

[1] In QCD, the long wavelength "magnetic" fluctuations have a mass of order g^2T, which is not large enough to prevent the breakdown of perturbation theory.

where m_B and g_B denote the bare mass and coupling constant, respectively. The upper bound of the integration over the imaginary time τ is $\beta = 1/T$, where T is the temperature. In line with the previous discussion, we introduce a specific renormalization scheme with the following, temperature dependent, renormalization conditions:

$$\begin{aligned} m^2 &= \Gamma^{(2)}(\mathbf{p}=\mathbf{0}, \omega=0, T), \\ 1 &= \frac{d\Gamma^{(2)}}{d\mathbf{p}^2}\big(\mathbf{p}^2=\mu^2, \omega=0, T\big), \\ g^2 &= \Gamma^{(4)}\big(\mathbf{p}_{sym}^2=\mu^2, \omega_i=0, T\big), \end{aligned} \tag{5}$$

where $\Gamma^{(2)}$ and $\Gamma^{(4)}$ are renormalized n-point functions, and $\mathbf{p}_{sym}^2$ refers to a symmetric combination of 3-momenta. There is of course a large flexibility in the choice of renormalization conditions. The scheme presented above satisfies the important requirement that the renormalized coupling constant g^2 becomes independent of the temperature when $\mu \gg T$, so that we can isolate unambiguously thermal effects when comparing theories with different values of the coupling constant. The determination of the mass is more subtle. We want to use the thermal mass m that enters the first renormalization condition (5) within the free propagators of the perturbative expansion. However, the renormalization condition does not determine m, it just fixes the finite parts of counterterms so that m has a prescribed value. In order to relate this value to a mass that is known, we shall calculate m_0, the mass at $T=0$, at the order of perturbation theory at which we work. This will then provide a self-consistent equation (occasionally referred to as a gap equation) for the determination of m as a function of m_0. One may be worried about the fact that the present scheme involves counterterms whose finite parts depend on the temperature. However, no temperature dependent infinities will remain if subdivergences are carefully eliminated.[2]

3. The 2-point function and the self-consistent thermal mass

The one-loop contribution to the 2-point function is easily calculated:

$$\Gamma^{(2)}(\mathbf{p}, \omega, T) = m^2 + \delta m^2 + \mathbf{p}^2 + \frac{g^2 T}{2}\sum_n \int \frac{d^d q}{(2\pi)^d}\frac{1}{\omega_n^2+\mathbf{q}^2+m^2}, \tag{6}$$

where $\omega_n = 2n\pi T$ is a Matsubara frequency, and we used the fact that the self-energy is independent of $\mathbf{p}^2$ in order to ignore the field renormalization factor. We have set $m_B^2 = m^2 + \delta m^2$, where m is the renormalized mass. Note that the renormalization of the coupling constant at one-loop order has an impact on the 2-point function only when this is calculated at 2-loop order (see next section). Accordingly, in Eq. (6), g is taken to be the renormalized coupling constant. It is convenient to set

$$I(m) \equiv T\sum_n \int_\mathbf{q} \frac{1}{\omega_n^2+\mathbf{q}^2+m^2} = \int_\mathbf{q} \frac{1+2n_\mathbf{q}}{2E_\mathbf{q}} \equiv I_0(m) + I_T(m), \tag{7}$$

where we have introduced the shorthand notation for momentum space integrations, to be used throughout this paper: $\int_\mathbf{q} = \int \frac{d^d q}{(2\pi)^d}$.

[2] The explicit calculations of the thermal mass and of the pressure that are presented in this paper, illustrate how the divergences are eliminated. The calculation of the zero temperature pressure in particular shows that, once the mass subdivergences are eliminated, the remaining divergence is a global divergence which is independent of the temperature, as it should (see Eq. (32)).