



Quantum black hole wave packet: Average area entropy and temperature dependent width



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ABSTRACT

A quantum Schwarzschild black hole is described, at the mini super spacetime level, by a non-singular wave packet composed of plane wave eigenstates of the momentum Dirac-conjugate to the mass operator. The entropy of the mass spectrum acquires then independent contributions from the average mass and the width. Hence, Bekenstein's area entropy is formulated using the $\langle \text{mass}^2 \rangle$ average, leaving the $\langle \text{mass} \rangle$ average to set the Hawking temperature. The width function peaks at the Planck scale for an elementary (zero entropy, zero free energy) micro black hole of finite rms size, and decreases Doppler-like towards the classical limit.

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Bekenstein–Hawking black hole area entropy [1] constitutes a triple point in the phase of physical theories, connecting gravity, quantum mechanics, and statistical mechanics. However, despite several illuminating derivations [2], the statistical roots of black hole entropy have not been fully revealed, not even at the level of discrete models [3]. There exist a few extreme black hole solutions [4], notably beyond general relativity, where one can apparently count micro states. But as far as the prototype Schwarzschild black hole is concerned, we still do not have the finest idea where these micro states are hiding, and how to enumerate them. A classical black hole is characterized by its event horizon, but once $\hbar$ is switched on (to allow for a finite Hawking temperature and non-zero Bekenstein entropy), even the innocent looking question ‘where is this horizon located’ lacks a decisive answer in the quantum or even in the semi-classical level.

The quantum-mechanical Schwarzschild black hole is hereby described by a non-singular minimal uncertainty wave packet composed of plane wave eigenstates. We carry out our analysis at the mini super spacetime level [5] without relying on theories beyond general relativity such as string theory [6], the fuzzball proposal [7], or loop quantum gravity [8] (see [9] for a different approach). Treating the black hole as a subsystem (a field theory defined on a black hole background is expected to be in a thermal state), its Gaussian mass spectrum becomes temperature dependent. We invoke Fowler prescription [10] for dealing with such

subsystems, and show that the associated statistical entropy acquires independent contributions from the average mass as well as from the width, and consistently formulate Bekenstein's area ansatz by means of the $\langle \text{mass}^2 \rangle$ average. While, as expected, the $\langle \text{mass} \rangle$ average turns out to be inversely proportional to Hawking temperature, a novel temperature dependent width function makes its appearance. The width function is maximal at the reduced Planck mass for an elementary quantum-mechanical black hole of finite rms size, for which both the entropy and free energy vanish and are minimal, and decreases Doppler-like towards the classical limit.

Let our starting point be the most general static radially symmetric line element, expressed in the form

$$ds^2 = -\frac{y(r)}{2r} dt^2 + \frac{2r}{x(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (1)$$

The unfamiliar x, y representation has been carefully designed to avoid the appearance of explicit r -dependence in the constrained Hamiltonian formalism (see Ref. [11] for the canonically transformed r -dependent Berry–Keating type [12] Hamiltonian). A tenable gauge pre-fixing option, namely defining a radial marker r whose geometrical interpretation is x, y -independent, has been harmlessly exercised. This has to be contrasted with the forbidden gauge prefixing of the ‘lapse’ function (the coefficient of dr^2 in this case), which kills the Hamiltonian constraint and introduces an unphysical degree of freedom (no gauge pre-fixing in Kuchar's midi superspace approach [13]). The more so at the mini super-spacetime level, where the general relativistic action $\int \mathcal{R} \sqrt{-g} d^4x$ is integrated out over time and solid angle into the mini action $\int \mathcal{L}(x, x', y, y', r) dr$.

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A word of caution is in order: Throughout this paper we treat $\int \mathcal{L}(q, q', r) dr$ in full mathematical analogy with $\int \mathcal{L}(q, \dot{q}, t) dt$. Technically, the t -evolution is traded for the r -evolution, both classically and quantum-mechanically, with the notions of Lagrangian and Hamiltonian being adapted accordingly. A similar technique has been introduced by York and Schmekel [14]. To sharpen the point, we clarify that our ‘Hamiltonian’ (to be identified with the momentum Dirac-conjugate [15] to the mass operator) has nothing to do with the physical mass of the black hole.

Up to a total derivative and an overall absorbable factor, the mini super-spacetime Lagrangian takes the form

$$\mathcal{L}(x, x', y, y') = \left(\frac{3x'}{4} - 2 \right) \sqrt{\frac{y}{x}} - \frac{y'}{4} \sqrt{\frac{x}{y}}. \quad (2)$$

Being linear in the ‘velocities’, it gives rise to two primary second class constraints, namely

$$\phi_y = p_y + \frac{1}{4} \sqrt{\frac{x}{y}} \approx 0, \quad \phi_x = p_x - \frac{3}{4} \sqrt{\frac{y}{x}} \approx 0, \quad (3)$$

whose Poisson brackets do not vanish $\{\phi_y, \phi_x\} = \frac{1}{2\sqrt{xy}}$. Following Dirac prescription [15], we are then driven from the naive Hamiltonian $\mathcal{H} = p_x x' + p_y y' - \mathcal{L} = 2\sqrt{y/x}$ to the total Hamiltonian

$$\mathcal{H}_T = 2\sqrt{\frac{y}{x}} + 2\frac{y}{x}\phi_y + 2\phi_x. \quad (4)$$

One can verify that the corresponding classical solution is (and is nothing but) the Schwarzschild solution

$$\frac{y(r)}{2\omega^2 r} = \frac{x(r)}{2r} = 1 - \frac{2m}{r}, \quad (5)$$

with no restrictions on the sign of the integration parameters m and ω . Along the classical trajectories the Hamiltonian takes the value $\mathcal{H} = 2\omega$, telling us that the $\mathcal{H}$ is not the total physical mass of the system.

To quantize the system it becomes crucial to calculate the Dirac brackets, and here one finds first of all

$$\{x, y\}_D = 2\sqrt{xy} \neq 0 \quad (6)$$

Counter-intuitively, and potentially with far reaching consequences, two metric components do not Dirac-commute. Moreover, the relation $\{x, \frac{1}{2}\mathcal{H}\}_D = 1$ paves then the way for the quantum-mechanical commutation relations $[x, \frac{1}{2}\mathcal{H}] = i\hbar$. $\mathcal{H}$ is then faithfully represented by

$$\mathcal{H} = -2i\hbar \frac{\partial}{\partial x}. \quad (7)$$

By the same token, in accord with Eq. (6), the other metric component y is represented by $y = \frac{1}{4}\mathcal{H}x\mathcal{H}$.

$\phi_{x,y}$ are second class constraints, so $\phi_x \psi = \phi_y \psi = 0$ are automatically fulfilled. Denoting by 2ω the eigenvalues of $\mathcal{H}$, the corresponding eigenstates are simple plane waves. Their full r -‘evolution’ is given by

$$\psi_\omega(x, r) = \frac{1}{\sqrt{4\pi}} e^{\frac{i}{\hbar}\omega(x-2r)}. \quad (8)$$

They are not localized and form a δ -normalizable set. The most general solution is of the form $\psi(x-2r)$. We are however after the ‘most classical’ wave packet defined by the minimal uncertainty relation $\Delta x \Delta \mathcal{H} = \hbar$, namely

$$\psi(x, r) = \frac{e^{-\frac{(x-2r+4m)^2}{64\sigma^2}}}{(2\pi)^{\frac{1}{4}} \sqrt{\sigma}}, \quad (9)$$

for which the classical Schwarzschild solution (5) is both the average as well as the most probable configuration. We thus expect the wave packet Eq. (9) to capture all semi-classical essence of black hole thermodynamics. We have limited ourselves in this paper to the ‘most classical’ black hole wave packet simply because Bekenstein–Hawking thermodynamics is formulated in the background of a classical event horizon. One can even construct an orthonormal tower of non-minimal uncertainty wave packets [11], to be regarded a prediction of the mini super-spacetime approach (to be discuss elsewhere), none of which sharing the Schwarzschild configuration as the most probable. Eq. (9) is a superposition of plane waves. Its Fourier transform is given by

$$\tilde{\psi}(\mathcal{H}) = \frac{2\sqrt{\sigma}}{(2\pi)^{\frac{1}{4}}} e^{-4\sigma^2 \mathcal{H}^2} e^{2im\mathcal{H}}. \quad (10)$$

We identify the mass operator as $M = \frac{1}{4}(2r - x)$ (in the $\mathcal{H}$ -language it reads $M = -\frac{1}{2}i\hbar \frac{\partial}{\partial \mathcal{H}}$). For the Gaussian wave packet (9) it means

$$\langle M \rangle = m, \quad \langle M^2 \rangle = m^2 + \sigma^2. \quad (11)$$

The black wave packet probability density $\psi^\dagger \psi$ can be directly translated into a statistical mechanics normalized mass spectrum

$$\rho(M; m, \sigma) = \frac{e^{-\frac{(M-m)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}. \quad (12)$$

While a non-negative average mass $m \geq 0$ (the classical choice) is soon to be dictated on thermodynamical grounds, the mass distribution must cover, for the sake of quantum completeness, the entire range $-\infty < M < \infty$. However, the probability to have negative masses drops like $\sim \exp(-m^2/2\sigma^2)$ towards the classical limit.

At this stage, one may wonder where is the black hole horizon actually located? As far as our wave packet is concerned, there is nothing special going on in the neighbourhood of $r = 2Gm/c^2$ (and actually also not near the origin). Supported by Eq. (6), this suggests that a sharp horizon is merely a classical gravitational concept. However, one may still effectively interpret Eq. (12) as the quantum-mechanical profile of the horizon, with a probability density $\rho(M; m, \sigma)$ to find it at radius $2Gm/c^2$. In some sense, this reminds us of the fuzzball proposal [7] where the black hole arises from coarse graining over horizon-free non-singular geometries. See [16] for horizon wave packets, and [14,17] for horizon fluctuations.

Treating the quantum-mechanical black hole as a subsystem (a field theory defined on a black hole background is expected to be in a thermal state whose temperature at infinity is the Hawking temperature), its Gaussian mass spectrum is temperature dependent. Following Fowler prescription for dealing with such a case, the naive partition function must be modified according to

$$Z(\beta) = \sum_n \rho_n e^{-\beta F_n(\beta)}, \quad (13)$$

with the Boltzmann factor being traded for the Gibbs–Helmholtz factor. The Helmholtz free energy function F obeys the Gibbs–Helmholtz equation

$$F + \beta \frac{\partial F}{\partial \beta} = E(\beta) \Rightarrow \beta F(\beta) = \int_{\beta_0}^{\beta} E(b) db, \quad (14)$$

with β_0 to be fixed on physical grounds. To proceed, it is convenient to discretize the problem by dividing the normal mass distribution $\rho(M; m, \sigma)$ into N equal probability and temperature

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