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Twisted invariances of noncommutative gauge theories

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ABSTRACT

We study noncommutative deformations of Yang–Mills theories and show that these theories admit a infinite, continuous family of twisted star-gauge invariances. This family interpolates continuously between star-gauge and twisted gauge transformations. The possible physical rôle of these start-twisted invariances is discussed.

1. Introduction

Noncommutative deformations of field theories have provided interesting workbenches where some properties of Quantum Field Theory can be probed (for reviews see [1]). Of particular interest in this field has been the study of gauge theories on noncommutative spaces. Apart from many other interesting features, these theories describe certain low-energy limit of string theory in the presence of a constant B-field background [2].

Although very interesting from a mathematical physics point of view, noncommutative deformation of Yang–Mills theories seem to have a limited phenomenological interest. In general, the deformation of the gauge transformations force the gauge fields to take values in the universal enveloping algebra of the gauge group [3]. In particular, it can be seen that the only gauge group for which the gauge transformations close is $\mathrm{U}(N)$ [4,5], thus excluding the phenomenologically more interesting special unitary groups (see however [6] for some proposals to realize nonunitary groups in noncommutative geometry). In addition to this, the theory suffers from instabilities at the quantum level. A one loop calculation of the dispersion relation for the noncommutative photon shows that it diverges at low momentum [7]. This instability can be removed by embedding the gauge theory into noncommutative $\mathcal{N}=4$ super-Yang–Mills theory at high energy, but at the price of introducing a severe fine tuning of the supersymmetry breaking scale [8]. Alternatively, lattice studies of noncommutative $\mathrm{U}(1)$ gauge theories have shown [9] that the tachyonic instability can be eliminated in a new nonperturbative phase of the theory characterized by the breaking of translation invariance.

In the construction of Yang–Mills theories on noncommutative spaces presented in [2] star-gauge transformations play a central rôle as the true gauge symmetry of the deformed theory [10]. This deformed symmetry acts in the standard way by (nonlocal) transformation of the fields. On the other hand, in [11,12] it was pointed out that the action of noncommutative Yang–Mills is also invariant under standard, i.e., commutative, gauge transformations provided the Leibniz rule is twisted accordingly. Although in this case the transformations are consistent for any gauge group, the equations of motion of the theory force now the gauge fields to take values on the universal enveloping algebra of the gauge group [12]. The study of these type of theories has attracted considerable attention [13].

The extra terms appearing in the twisted Leibniz rule in this type of theories can be understood as due to a transformation of the star-product itself under gauge transformations [14]. From this point of view twisted gauge transformations are not standard, *bona-fide* transformations since they involve not only the transformation of fields but of the product operation as well. This prevents a direct application of the standard procedures to obtain Noether currents and/or Ward identities associated with these symmetries. It is important to keep in mind that twisted gauge theories are also invariant under the corresponding star-gauge transformation, which is a standard symmetry of the theory acting only on fields. In [14] it was argued that star-gauge transformations play a custodial rôle in guaranteeing the existence of conserved current and Ward identities. This point of view was further supported in [15], where it was argued that the consistency of the twisted gauge theory requires the presence of the custodian star-gauge symmetry.

The crucial point in the construction of noncommutative twisted gauge theories is the realization that the product used in writing the action and the product involved in the gauge transformations does not have to be the same if other conditions like the Leibniz rule are relaxed. In the case of Refs. [11,12] gauge transformations act through the ordinary, commutative, product, whereas the action is con-

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structed in terms of the star-product. In this note we show that this construction can be generalized in such a way that noncommutative gauge theories can be shown to be invariant under star-gauge transformations defined with *any* noncommutative parameter, with the appropriate twist of the Leibniz rule. This family of invariances continuously interpolate between star-gauge symmetry and twisted gauge transformations defined in terms of the standard commutative product.

2. Star-twisted gauge transformations: Heuristic derivation

Pure noncommutative Yang–Mills. Let us consider the algebra $\mathcal{A}$ of functions on $\mathbb{R}^d$ and the Groenewold–Moyal star-product between elements of this algebra defined as

$$f(x)\star_\theta g(x)\equiv f(x)\exp\left[\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu\right]g(x). \tag{2.1}$$

For the sake of clarity, here and in the following we always denote explicitly the noncommutativity parameter used in the definition of the star-product. Pure gauge theories on noncommutative spaces can be constructed in terms of the previous noncommutative product by [1]

$$S=-\frac{1}{2g^2}\int d^dx\,\mathrm{tr}\big(F_{\mu\nu}\star_\theta F^{\mu\nu}\big),\qquad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-i[A_\mu,A_\nu]_\theta, \tag{2.2}$$

where we have used the obvious notation $[A,B]_\theta\equiv A\star_\theta B-B\star_\theta A$. This action is invariant under the star-gauge symmetry

$$\delta^\theta_\varepsilon A_\mu=\partial_\mu\varepsilon+i[\varepsilon,A_\mu]_\theta. \tag{2.3}$$

Because of this deformed gauge transformation the gauge group has to be restricted to U(N). For any other gauge group G, Eq. (2.3) forces the gauge field A_μ has to take values on the universal enveloping algebra of the Lie algebra of G. Here we confine our analysis to theories with gauge group U(N).

The idea of Ref. [11,12] is that the action (2.2) can also be made invariant under standard (undeformed) gauge transformations

$$\delta^0_\varepsilon A_\mu=\partial_\mu\varepsilon+i[\varepsilon,A_\mu]_0\equiv\partial_\mu\varepsilon+i(\varepsilon\cdot A_\mu-A_\mu\cdot\varepsilon), \tag{2.4}$$

provided the action of the transformations on the products of fields is changed appropriately

$$\begin{aligned}\delta^0_\varepsilon(A_\mu\star_\theta A_\nu)=\sum_{n=0}^\infty\frac{(-i/2)^n}{n!}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}\cdots\theta^{\alpha_n\beta_n}\Big\{&\big(\big[\partial_{\alpha_1},\big[\partial_{\alpha_2},\ldots\big[\partial_{\alpha_n},\delta^0_\varepsilon\big]\ldots\big]\big]A_\mu\big)\star_\theta(\partial_{\beta_1}\partial_{\beta_2}\cdots\partial_{\beta_n}A_\nu)\\&+(\partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_n}A_\mu)\star_\theta\big(\big[\partial_{\beta_1},\big[\partial_{\beta_2},\ldots\big[\partial_{\beta_n},\delta^0_\varepsilon\big]\ldots\big]\big]A_\nu\big)\Big\}.\end{aligned} \tag{2.5}$$

In this series the term $n=0$ gives the standard Leibniz rule which is corrected by an infinite number of terms with arbitrary number of derivatives. This transformation of the product of two gauge fields implies that the field strength transforms as

$$\delta^0_\varepsilon F_{\mu\nu}=[i\varepsilon,F_{\mu\nu}]_0\equiv i(\varepsilon\cdot F_{\mu\nu}-F_{\mu\nu}\cdot\varepsilon). \tag{2.6}$$

This transformation, together with the twisted Leibniz rule, guarantees the invariance of the action under twisted gauge transformations.

Let us now go back to the action (2.2) but consider a star-gauge transformation with parameter $\theta'^{\mu\nu}\neq\theta^{\mu\nu}$

$$\delta^{\theta'}_\varepsilon A_\mu=\partial_\mu\varepsilon+i[\varepsilon,A_\mu]_{\theta'}. \tag{2.7}$$

The variation of the field strength $F_{\mu\nu}$ in Eq. (2.2) under this transformation can be written as

$$\delta^{\theta'}_\varepsilon F_{\mu\nu}=[i\varepsilon,\partial_\mu A_\nu-\partial_\nu A_\mu]_{\theta'}+i[\partial_\mu\varepsilon,A_\nu]_{\theta'}-i[\partial_\nu\varepsilon,A_\mu]_{\theta'}-i\delta^{\theta'}_\varepsilon[A_\mu,A_\nu]_\theta. \tag{2.8}$$

In order to evaluate the last term explicitly we need to compute the action of the θ'-star gauge transformation on the θ-star product, $\delta^{\theta'}_\varepsilon(A_\mu\star_\theta A_\nu)$. For this we use the deformed Leibniz rule

$$\begin{aligned}\delta^{\theta'}_\varepsilon(A_\mu\star_\theta A_\nu)=\sum_{n=0}^\infty&\frac{(-i/2)^n}{n!}\big(\theta^{\alpha_1\beta_1}-\theta'^{\alpha_1\beta_1}\big)\big(\theta^{\alpha_2\beta_2}-\theta'^{\alpha_2\beta_2}\big)\cdots\big(\theta^{\alpha_n\beta_n}-\theta'^{\alpha_n\beta_n}\big)\\&\times\Big\{\big(\big[\partial_{\alpha_1},\big[\partial_{\alpha_2},\ldots\big[\partial_{\alpha_n},\delta^{\theta'}_\varepsilon\big]\ldots\big]\big]A_\mu\big)\star_\theta(\partial_{\beta_1}\partial_{\beta_2}\cdots\partial_{\beta_n}A_\nu)\\&+(\partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_n}A_\mu)\star_\theta\big(\big[\partial_{\beta_1},\big[\partial_{\beta_2},\ldots\big[\partial_{\beta_n},\delta^{\theta'}_\varepsilon\big]\ldots\big]\big]A_\nu\big)\Big\}.\end{aligned} \tag{2.9}$$

After a tedious but straightforward calculation one arrives at

$$\delta^{\theta'}_\varepsilon(A_\mu\star_\theta A_\nu)=(\partial_\mu\varepsilon)\star_{\theta'}A_\nu+A_\mu\star_{\theta'}(\partial_\nu\varepsilon)+[i\varepsilon,A_\mu\star_\theta A_\nu]_{\theta'}. \tag{2.10}$$

In getting this expression we have to use two identities valid for any pair of functions $f(x)$, $g(x)$. The first equality is

$$[f,\partial_{\mu_1}\cdots\partial_{\mu_n}g]_{\theta'}=\sum_{k=0}^n(-1)^{n-k}\partial_{(\mu_1}\cdots\partial_{\mu_k}[\partial_{\mu_{k+1}}\cdots\partial_{\mu_n)}f,g]_{\theta'}, \tag{2.11}$$

where the parenthesis indicates the symmetrization

$$\partial_{(\mu_1}\cdots\partial_{\mu_k}[\partial_{\mu_{k+1}}\cdots\partial_{\mu_n)}f,g]_{\theta'}\equiv\sum_{\sigma\in S_n}\frac{1}{k!(n-k)!}\partial_{\sigma(\mu_1}\cdots\partial_{\mu_k}[\partial_{\mu_{k+1}}\cdots\partial_{\mu_n)}f,g]_{\theta'} \tag{2.12}$$