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A semiclassical kinetic theory of Dirac particles and Thomas precession

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ABSTRACT

Kinetic theory of Dirac fermions is studied within the matrix valued differential forms method. It is based on the symplectic form derived by employing the semiclassical wave packet build of the positive energy solutions of the Dirac equation. A satisfactory definition of the distribution matrix elements imposes to work in the basis where the helicity is diagonal which is also needed to attain the massless limit. We show that the kinematic Thomas precession correction can be studied straightforwardly within this approach. It contributes on an equal footing with the Berry gauge fields. In fact in equations of motion it eliminates the terms arising from the Berry gauge fields.

1. Introduction

Dirac equation which describes massive spin-1/2 particles possesses either positive or negative energy solutions described by 4-dimensional spinors. However, to furnish a well defined one particle interpretation, instead of employing both types of solutions, a wave packet build of only positive energy plane wave solutions should be preferred. A nonrelativistic semiclassical dynamics can be obtained employing this wave packet. Semiclassical limit may be useful to have a better understanding of some quantum mechanical phenomena. In the massless limit Dirac equation leads to two copies of Weyl particles which possess opposite chirality. Recently, the chiral semiclassical kinetic theory has been formulated to embrace the anomalies due to the external electromagnetic fields in $3+1$ dimensions [1,2]. This remarkable result was extended to any even dimensional space–time by making use of differential forms in [3] by introducing some classical variables corresponding to spin. Although in [3] non-Abelian anomalies have been incorporated into the particle currents, the solutions of phase space velocities in terms of phase space variables were missing. In [4] a complete description of the chiral semiclassical kinetic theory in any even dimension was established by introducing a symplectic two-form which is a matrix labeled with "spin indices", without introducing any classical variable corresponding to spin degrees of freedom. In this formalism, although the classical phase space variables are the ordinary ones, the velocities arising from the equations of motion are matrix valued. It has been shown in [5] that this matrix valued symplectic two form can be derived within the semiclassical wave packet formalism [6,7]. In fact the "spin indices" label the linearly independent positive energy solutions.

In the formulation of the Hamiltonian dynamics starting with a first order Lagrangian, the related Hamiltonian should be provided. In the development of the chiral kinetic theory the Hamiltonian was taken as the positive relativistic energy of the free Weyl Hamiltonian. However, later it was shown that the adequate Hamiltonian should contain all the first order terms in Planck constant [8] which can be attained by employing the method introduced in [9]. Independently, the same Hamiltonian was conjectured in [10] to restore the Lorentz invariance of the semiclassical chiral theory. To obtain it one first has to derive the Hamiltonian of the massive spin-1/2 fermion and then take the massless limit.

Massive fermions also appear in condensed matter systems which were studied in terms of wave packets in [11,12]. The semiclassical kinetic theory of Dirac particles was also discussed in [13], where the Berry gauge fields have been given in a different basis and some classical degrees of freedom have been assigned to spin. We would like to establish the semiclassical kinetic theory of Dirac particles within the formalism given in [4]. There are some advantages of employing this method. First of all because of not attributing any classical variables to spin but taking them into account by considering quantities matrix valued in "spin space", the calculations can be done explicitly. The differential forms method provides us the solutions of the equations of motion for the phase space velocities in terms of phase space variables straightforwardly. Thus the particle currents can be readily derived. Moreover, we will show that within this formalism one can study the relativistic correction known as Thomas precession [14].

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Thomas precession stems from the fact that a Lorentz boost can be written as two successive Lorentz boosts accompanied by a rotation which is called Thomas rotation. This purely kinematic phenomenon is essential to obtain the classical evolution of electron's spin correctly, without referring to the Dirac equation. Thomas precession should also contribute to the equations of motion of phase space variables. In fact, due to Thomas precession the covariant formalisms of Dirac particles yield equations of motion where anomalous velocity terms do not emerge [15,16]. However, as we will see the equations of motion derived within the wave packet formalism possess anomalous velocity terms arising from the Berry curvature. This would have been expected because of the fact that our nonrelativistic formalism is not aware of Thomas rotation. Correction due to Thomas rotation should be installed in the formalism. We will show that our formalism suites well to take this correction into account: It contributes to the one-form obtained by the semiclassical wave packet on an equal footing with the Berry gauge field. In fact, it yields the cancellation of the anomalous velocity terms ignoring the higher order terms in momentum. The connection of Berry gauge fields and Thomas precession was first observed by Mathur [17].

In Section 2 the one-form corresponding to the first order Lagrangian is obtained by the wave packet composed of the positive energy plane wave solutions of the Dirac equation. Then the related symplectic form is constructed and the solutions of the equations of motion for the velocities of the Dirac particle are established in Section 3. Spin degrees of freedom are taken into account by letting the velocities be matrix valued. So that, one should consider a matrix valued distribution function. In contrary to spin, helicity operator is a conserved quantity for the free Dirac particle. Moreover, we would like to split the particles as right-handed and left-handed appropriate to consider the massless limit and the chiral currents when there is an imbalance of chiral particles. Therefore we introduce a change of basis to the helicity basis as clarified in Section 4. Employing distribution matrix in the adequate basis we then can write the particle number density and the related current by the velocities written in terms of the phase space variables in Section 3. In Section 5 we obtained the massless limit by constructing the helicity eigenstates explicitly. In Section 6 a brief review of Thomas rotation is presented. Then, we presented how it appears in the one-form obtained by the semiclassical wave packet. We will see that up to higher order terms in momentum it contributes as the Berry gauge field but with an opposite sign. Our semiclassical formalism should be supported by an equation governing spin dynamics. This will be attained employing Gosselin–Berárd–Mohrbach (GBM) method [9] which is also needed to derive the semiclassical Hamiltonian. In the last section the results obtained and some possible applications are discussed. Moreover, we clarified in which frame we obtained the semiclassical theory.

2. Semiclassical wave packet

Dirac particle interacting with the external electromagnetic fields $\boldsymbol{\mathcal{E}}$, $\boldsymbol{B}$, whose vector and scalar potentials are $\boldsymbol{a}(\boldsymbol{x})$ and $a_0(\boldsymbol{x})$, is described by the Dirac Hamiltonian $H = H_0 + ea_0(x)$, where

$$H_0(\boldsymbol{p} - e\boldsymbol{a}(\boldsymbol{x})) = \beta m + \boldsymbol{\alpha}\cdot(\boldsymbol{p} - e\boldsymbol{a}(\boldsymbol{x})). \tag{1}$$

We set the speed of light $c = 1$ and let $e < 0$ for electron. The representation of the α_i; $i = 1, 2, 3$, and β matrices are chosen as

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where σ_i are the Pauli spin matrices.

A semiclassical formulation given in terms of the wave packet made of the positive energy solutions of the Dirac equation provides a well defined one particle interpretation. Let the position of the wave packet center in the coordinate space be $\boldsymbol{x}_c$, and the corresponding momentum be $\boldsymbol{p}_c$. The semiclassical wave packet is defined in terms of the positive energy solutions, $u^\alpha(\boldsymbol{x}, \boldsymbol{p})$; $\alpha = 1, 2$, as

$$\psi_{\boldsymbol{x}}(\boldsymbol{p}_c, \boldsymbol{x}_c) = \sum_\alpha \xi_\alpha u^\alpha(\boldsymbol{p}_c, \boldsymbol{x}_c) e^{-i\boldsymbol{p}_c\cdot\boldsymbol{x}/\hbar}.$$

For simplicity we deal with constant ξ_α coefficients. We would like to attain the one-form η which is defined through dS as

$$dS \equiv \int [dx]\delta(\boldsymbol{x}_c - \boldsymbol{x})\Psi^\dagger_{\boldsymbol{x}}(-i\hbar d - H_{0D}dt)\,\Psi_{\boldsymbol{x}} = \sum_{\alpha\beta} \xi^*_\alpha \eta^{\alpha\beta}\xi_\beta.$$

H_{0D} is the block diagonal Hamiltonian which should be derived from (1). Now, calculating dS one attains the η one-form as follows.

$$\eta^{\alpha\beta} = -\delta^{\alpha\beta}\boldsymbol{x}_c\cdot d\boldsymbol{p}_c - \boldsymbol{a}^{\alpha\beta}\cdot d\boldsymbol{x}_c - \boldsymbol{A}^{\alpha\beta}\cdot d\boldsymbol{p}_c - H^{\alpha\beta}_{0D}dt. \tag{2}$$

Here we introduced the matrix valued Berry gauge fields

$$\boldsymbol{a}^{\alpha\beta} = i\hbar u^{\dagger(\alpha)}(\boldsymbol{p}_c, \boldsymbol{x}_c)\frac{\partial}{\partial \boldsymbol{x}_c}u^{(\beta)}(\boldsymbol{p}_c, \boldsymbol{x}_c),$$
$$\boldsymbol{A}^{\alpha\beta} = i\hbar u^{\dagger(\alpha)}(\boldsymbol{p}_c, \boldsymbol{x}_c)\frac{\partial}{\partial \boldsymbol{p}_c}u^{(\beta)}(\boldsymbol{p}_c, \boldsymbol{x}_c).$$

Although we deal with the $(3+1)$-dimensional space–time, the derivation of η does not depend on dimension.

3. Semiclassical Hamiltonian dynamics

Instead of solving the Dirac equation in the presence of the electromagnetic vector potential $\boldsymbol{a}(\boldsymbol{x})$, we substitute $\boldsymbol{p} \to \boldsymbol{p} + e\boldsymbol{a}(\boldsymbol{x})$, in (1) and consider the free particle solutions with $E = \sqrt{p^2 + m^2}$. Then the positive energy solutions will not possess $\boldsymbol{x}$ dependence. Therefore by renaming $(\boldsymbol{x}_c, \boldsymbol{p}_c) \to (\boldsymbol{x}, \boldsymbol{p})$ and setting $\boldsymbol{a}^{\alpha\beta} = 0$, we obtain the one-form

$$\eta = p_i dx_i + ea_i dx_i - A_i dp_i - H dt. \tag{3}$$

The repeated indices are summed over. We suppress the matrix indices α, β, and do not write explicitly the unit matrix $\mathbb{I}$, when it is not necessary. We deal with the Hamiltonian $H = H_D(\boldsymbol{p}) + ea_0(\boldsymbol{x})$, where

$$H_D(\boldsymbol{p}) = E - \hbar e\left(m\frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2E^2} + \frac{(\boldsymbol{B}\cdot\boldsymbol{p})(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{2E^2(E+m)}\right). \tag{4}$$

This semiclassical Hamiltonian includes all contributions which are at the first order in $\hbar$. It is attained by making use of the GBM method [9].

To obtain Hamiltonian dynamics we have to introduce a symplectic two-form. In general the constituents of the one-form (matrix) η can be non-Abelian, so that we adopt the definition of the symplectic two-form $\tilde{\omega}$ to be

$$\tilde{\omega} = d\eta - \frac{i}{\hbar}\eta\wedge\eta.$$

We would like to emphasize that it is a matrix in spin indices. Employing the one-form (3), it yields

$$\tilde{\omega} = dp_i \wedge dx_i + e\mathcal{E}_i dx_i \wedge dt + f_i dp_i \wedge dt - G + eF.$$

Here, $\mathcal{E}_i = -\left(\frac{\partial a_i}{\partial t} + \frac{\partial a_0}{\partial x_i}\right)$, is the electric field and