Non-commutative and commutative vacua effects in a scalar torsion scenario

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ABSTRACT

In this work, the effects of non-commutative and commutative vacua on the phase space generated by a scalar field in a scalar torsion scenario are investigated. For both classical and quantum regimes, the commutative and non-commutative cases are compared. To take account the effects of non-commutativity, two well known non-commutative parameters, θ and β, are introduced. It should be emphasized, the effects of β which is related to momentum sector has more key role in comparison to θ which is related to space sector. Also the different boundary conditions and mathematical interpretations of non-commutativity are explored.

1. Introduction

In this work our aim is studying a non-commutative model of scalar torsion gravity.

Recently some astrophysical observations have shown that the Universe undergoing an accelerated phase era. To justify this unexpected result, scientists have proposed some different models such as, scalar field models [1–4] and modify theories of gravity [5–8]. For the latter proposal one can deal with teleparallel equivalent of general relativity [9–12], in which the field equations are second order [13]. In addition in this scenario the Levi-Civita connections replaced by Weitzenböck connection, where has no curvature but only torsion [14].

It is obvious that for the first time, the non-commutative formalism between the space–time coordinate was introduced by [15]. Also the geometrical concept based on this model recently attracts more interesting namely non-commutative geometry [16–19]. It is notable the recent investigations of string theory, supersymmetry, M-theory and so on [20,21], motivated scientists to study classical and quantum cosmology in such frame. The effects of non-commutativity in cosmology have been investigated by two well-known models, i.e. minisuperspace [22,23] and phase space [24], while the geometrical structure of the underlying space–time unchanged [25]. In this work our means is that to build up a non-commutative scenario by means of a deformation achieved by Moyal product [26], for a scalar torsion gravity [27] in both classical and quantum levels. Although the non-minimal coupling term has a richer structure and experiencing the phantom-divide crossing and so on [9,28,29], but we want to consider the simplest form, minimal quintessence-like, of a scalar torsion scenario in comparison to standard quintessence scenario for this investigation.

The organization of this work is asfollows. In Section 2, a brief review about scalar $f(T)$ gravity cosmology and general properties of the model are discussed. In Section 3, the results of our investigations for scalar torsion gravity are discussed in classical level for both commutative and non-commutative frames. The Section 4, is devoted to the same details of Section 3 but in quantum level. And at last the Section 5, is concerned with the conclusion and discussion.

2. General framework

The teleparallel theory of gravity is defined in the Weitzenböck's space–time by the following line element

$$dS^2 = N^2 dt^2 - a^2(t)\delta_{ij}dx^i dx^j, \tag{1}$$

where N is the lapse function. Also it is considerable that, theory can be described in the tangent space, which allows us to rewrite the line element (1) as

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$$dS^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ij}\theta^i\theta^j, \qquad (2)$$

$$dx^\mu = e_i{}^\mu\theta^i, \quad \theta^i = e^i{}_\mu dx^\mu, \qquad (3)$$

where $\eta_{ij} = \text{diag}[1, -1, -1, -1]$ and $e_i{}^\mu e^i{}_\nu = \delta^\mu_\nu$ or $e_i{}^\mu e^j{}_\mu = \delta^j_i$, and the matrix $e^a{}_\mu$ are called tetrads that indicate the dynamic fields of the theory.

According to theses fields, the Weitzenböck's connection is defined as

$$\Gamma^\alpha_{\mu\nu} = e_i{}^\alpha\partial_\nu e^i{}_\mu = -e^i{}_\mu\partial_\nu e_i{}^\alpha, \qquad (4)$$

that to be used for construction the main geometrical objects of the space–time. The components of the tensor torsion and the contorsion are defined respectively as

$$T^\rho{}_{\mu\nu} \equiv e_l{}^\rho\left(\partial_\mu e^l{}_\nu - \partial_\nu e^l{}_\mu\right), \qquad (5)$$

$$K^{\mu\nu}{}_\rho \equiv -\frac{1}{2}\left(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}\right). \qquad (6)$$

It was defined a new tensor $S_\rho{}^{\mu\nu}$, to obtain the scalar equivalent to the curvature scalar of general relativity, i.e. Ricci scalar, that is as

$$S_\rho{}^{\mu\nu} \equiv \frac{1}{2}\left(K^{\mu\nu}{}_\rho + \delta^\mu_\rho T^{\alpha\nu}{}_\alpha - \delta^\nu_\rho T^{\alpha\mu}{}_\alpha\right). \qquad (7)$$

Hence, the torsion scalar is defined by the following contraction

$$T \equiv S_\rho{}^{\mu\nu}T^\rho{}_{\mu\nu}. \qquad (8)$$

In studying the scalar torsion model instead of non-minimal coupling scenario [9,29], the minimal coupling action of the theory is defined by generalizing the teleparallel theory, as [27]

$$\mathcal{A} = \int d^4x|e|\left[\xi T - \zeta\frac{1}{2}\eta^{ij}e_i{}^\mu e_j{}^\nu\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right], \qquad (9)$$

where $|e| = \sqrt{-g}$ and T is the torsion scalar, ξ and ζ are constant. Let us choose the following set of diagonal tetrads related to the metric (1) as

$$[e^a{}_\mu] = \text{diag}[N, a, a, a], \qquad (10)$$

the determinant of the matrix (10) is $e = Na^3$. The components of the torsion tensor (5) for the tetrads (10) are given by

$$T^1{}_{01} = \frac{\dot{a}}{Na} = T^2{}_{02} = T^3{}_{03}, \qquad (11)$$

and the components of the corresponding contorsion are

$$K^{01}{}_1 = \frac{\dot{a}}{Na} = K^{02}{}_2 = K^{03}{}_3. \qquad (12)$$

The components of the tensor $S_\alpha{}^{\mu\nu}$, in (7), are given by

$$S_1{}^{10} = \left(\frac{\dot{a}}{Na}\right) = S_2{}^{20} = S_3{}^{30}. \qquad (13)$$

By using the components (11) and (13), the torsion scalar (8) is given by

$$T = -6\frac{\dot{a}^2}{(Na)^2}. $$

Substituting Eq. (10) into the action (9) the Lagrangian density can be achieved as follows

$$\mathcal{L} = Na^3\left(-6\xi\frac{\dot{a}^2}{(Na)^2} + \frac{\zeta}{2N^2}\dot{\phi}^2 - V(\phi)\right). \qquad (14)$$

For more convenience the above constants ξ and ζ can be considered as $\xi = 1/6$, $\zeta = 1/2$. Using a new set of variables,

$$x = \frac{a^2}{2}\cosh\phi, \quad y = \frac{a^2}{2}\sinh\phi, \qquad (15)$$

where $a^2 = 2(x - y)e^\phi$, one can rewrite the above Lagrangian density as follows

$$\mathcal{L} = (\dot{y}^2 - \dot{x}^2) - 4(x - y)e^\phi V(\phi). \qquad (16)$$

Thence, the corresponding Hamiltonian density is

$$\mathcal{H} \equiv \sum_\alpha \dot{x}^\alpha\frac{\partial\mathcal{L}}{\partial\dot{x}^\alpha} - \mathcal{L} = \frac{1}{2}\left(\frac{1}{2}P_y^2 - \frac{1}{2}P_x^2\right) + 4(x - y)e^\phi V(\phi), \qquad (17)$$

where $V(\phi) = 2V_0\exp[-\phi]$ and V_0 is a constant.

3. The cosmological evolution in classical regime

It is clear the classical solutions of a specific Hamiltonian can be easily yielded. However we want to inspect the effects of non-commutativity in classical level, then compare our results with commutative case.

3.1. Commutative algebra

It is well known the Poisson brackets between components of the classical phase space variables are as

$$\{x_i, x_j\} = \{p_i, p_j\} = 0, \qquad \{x_i, p_j\} = \delta_{ij}, \qquad (18)$$

where $x_i(i = 1, 2) = x, y$ and $p_i(i = 1, 2) = p_x, p_y$. Assuming $N = 1/a$, the equations of motion to be as

$$\dot{x} = \{x, \mathcal{H}\} = -\frac{p_x}{2}, \qquad \dot{p_x} = \{p_x, \mathcal{H}\} = -8V_0, \qquad (19)$$

$$\dot{y} = \{y, \mathcal{H}\} = \frac{p_y}{2}, \qquad \dot{p_y} = \{p_y, \mathcal{H}\} = 8V_0. \qquad (20)$$

Integrating the above equations, get

$$x(t) = 4V_0t^2 - p_{0x}t + x_0, \qquad p_x(t) = -8V_0t + p_{0x} \qquad (21)$$

$$y(t) = 4V_0t^2 + p_{0y}t + y_0, \qquad p_y(t) = 8V_0t + p_{0y}, \qquad (22)$$

where x_0, y_0, p_{0x} and p_{0y} are integration constants. In addition the constraint equation between them, by using the zero energy condition, $\mathcal{H} \equiv 0$, yields

$$p_{0x}^2 - p_{0y}^2 = -16V_0(y_0 - x_0). \qquad (23)$$

It is clear the Eqs. (21) and (22) have the same form of the equation motion of a particle with a constant acceleration. one can apply the condition $x > 0$, with the bound $p_{0x}^2 - 16V_0x_0 < 0$ to obtain the constraint $p_{0y}^2 - 16V_0y_0 < 0$ from relation (23), which indicates that $y > 0$. So only half of minisuperspace $x > y > 0$ is covered by dynamical variables. The evolution of scale factor and scalar field by combination Eqs. (15), (21) and (22) are given as follows

$$a(t) = \left(8|p_{0x}|(8V_0t^3 + 2x_0t)\right)^{1/4}, \qquad (24)$$

$$\phi(t) = \frac{1}{2}\ln\left(\frac{8V_0t^2 + 2x_0}{2|p_{0x}|t}\right), \qquad (25)$$

where we suppose $x_0 = y_0$ and $p_{0x} = p_{0y}$, in agreement with Eq. (23). Based on Eq. (24), $\ddot{a} < 0$ so the Universe is in a decelerated phase epoch. According to the Eq. (15), one can define an effective scale factor, $a_{eff}^2 = a^2e^{-\phi}$, which is equal to

$$a_{eff}^2 = 2(x - y) = 4|p_{0x}|t. \qquad (26)$$