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Rotating quantum states

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ABSTRACT

We revisit the definition of rotating thermal states for scalar and fermion fields in unbounded Minkowski space–time. For scalar fields such states are ill-defined everywhere, but for fermion fields an appropriate definition of the vacuum gives thermal states regular inside the speed-of-light surface. For a massless fermion field, we derive analytic expressions for the thermal expectation values of the fermion current and stress–energy tensor. These expressions may provide qualitative insights into the behaviour of thermal rotating states on more complex space–time geometries.

1. Introduction

In the canonical quantisation of a free field, an object of fundamental importance is the vacuum state, from which states containing particles are constructed. For fields of all spins, the process starts by expanding the classical field in terms of an orthonormal basis of field modes, which are split into positive and negative frequency modes. The expansion coefficients are promoted to operators, the expansion coefficients of the positive frequency modes being particle annihilation operators.[1] The vacuum state is defined as the state annihilated by all the particle annihilation operators. The definition of a vacuum state is therefore dependent on how the field modes are split into positive and negative frequency modes. This split is restricted for a quantum scalar field by the fact that positive frequency modes must have positive Klein–Gordon norm. For a quantum fermion field, both positive and negative frequency fermion modes have positive Dirac norm, so the split of the field modes into positive and negative frequency is less constrained compared with the scalar field case. There is therefore more freedom in how the vacuum state is defined for a fermion field, leading to more freedom in how states containing particles are defined.

In this letter we explore this difference between scalar and fermion quantum fields by considering the definition of rotating vacuum and thermal states in Minkowski space. This toy model reveals that there are quantum states which can be defined for a fermion field but which have no analogue for scalar fields.

2. Rotating scalars

We consider Minkowski space in cylindrical coordinates $(t_{\text{Mink}}, \rho, \varphi_{\text{Mink}}, z)$.[2] We wish to define quantum states which are rigidly rotating with angular velocity $\mathbf{\Omega}$. Choosing the z axis of the coordinate system along the angular velocity vector $\mathbf{\Omega}$, the line element of the rotating space–time can be found by making the transformation $\varphi = \varphi_{\text{Mink}} - \Omega t_{\text{Mink}}$, $t = t_{\text{Mink}}$ in the usual Minkowski line element, giving:

$$ds^2 = -\left(1 - \rho^2\Omega^2\right) dt^2 + 2\rho^2\Omega\, dt\, d\varphi + d\rho^2 + \rho^2\, d\varphi^2 + dz^2. \tag{1}$$

The Killing vector ∂_t, which defines the co-rotating Hamiltonian $H = i\partial_t$, becomes null on the speed-of-light surface (SOL), defined as the surface where $\rho = \Omega^{-1}$. The Klein–Gordon equation for a scalar field of mass μ on the space–time (1) is:

$$\left[-(H + \Omega L_z)^2 + \frac{L_z^2}{\rho^2} + P_z^2 - \partial_\rho^2 - \frac{\partial_\rho}{\rho} + \mu^2\right]\Phi(x) = 0, \tag{2}$$

where $P_z = -i\partial_z$ and $L_z = -i\partial_\varphi$ are the z components of the momentum and angular momentum operators, respectively. The mode solutions of (2) are:

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[1] The adjoints of the expansion coefficients of the negative frequency modes are also particle annihilation operators. For a real scalar field, these annihilation operators are the same as the expansion coefficients of the positive frequency modes; for a fermion field they are different.

[2] Throughout this paper we use units in which $c = \hbar = k_B = 1$.

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$$f_{\omega km}(x) = \frac{1}{\sqrt{8\pi^2|\omega|}} e^{-i\widetilde{\omega}t + im\varphi + ikz} J_m(q\rho), \tag{3}$$

where $J_m(q\rho)$ is the Bessel function of the first kind of order m, m is the eigenvalue of L_z, k is the eigenvalue of P_z, q is the longitudinal component of the momentum and $\omega = \pm\sqrt{\mu^2 + q^2 + k^2}$ gives the Minkowski energy of the mode. The eigenvalue of the Hamiltonian, $\widetilde{\omega} = \omega - \Omega m$, represents the energy of the mode as seen by a co-rotating observer. It is convenient to introduce the shorthand $j = (\omega_j, k_j, m_j)$ and

$$\delta(j, j') = \delta_{m_j m_{j'}} \delta(k_j - k_{j'}) \frac{\delta(\omega_j - \omega_{j'})}{|\omega_j|}. \tag{4}$$

Using the Klein–Gordon inner product:

$$\langle f, g\rangle = -i \int d^3x \sqrt{-g} \left(f^* \partial^t g - g \partial^t f^*\right), \tag{5}$$

the norm of the modes (3) can be calculated:

$$\langle f_j, f_{j'}\rangle = \frac{\omega_j}{|\omega_j|} \delta(j, j'). \tag{6}$$

As discussed by Letaw and Pfautsch [1], particles must be described by modes with positive norm ($\omega_j > 0$), implying the following expansion for the scalar field operator:

$$\Phi(x) = \sum_{m_j=-\infty}^{\infty} \int_{\mu}^{\infty} \omega_j \, d\omega_j \int_{-p_j}^{p_j} dk_j \left[f_j(x) a_j + f_j^*(x) a_j^\dagger\right], \tag{7}$$

where $p_j = \sqrt{q_j^2 + k_j^2}$ is the Minkowski momentum. The one-particle annihilation and creation operators, a_j and $a_j^\dagger$, satisfy the canonical commutation relations $[a_j, a_{j'}^\dagger] = \delta(j, j')$. The induced vacuum state $|0\rangle$, satisfying $a_j|0\rangle = 0$, coincides with the Minkowski vacuum [1].

At finite inverse temperature $\beta = T^{-1}$, Vilenkin [2] gives the following thermal expectation value (t.e.v.):

$$\langle a_j^\dagger a_{j'}\rangle_\beta = \frac{\delta(j, j')}{e^{\beta\widetilde{\omega}_j} - 1}. \tag{8}$$

The above expression cannot hold when $\widetilde{\omega}_j < 0$ [2], since it would imply that the vacuum expectation value of $a_j^\dagger a_{j'}$, obtained by taking the limit $\beta \to \infty$, is non-zero, contradicting the definition of the vacuum. Furthermore, the divergent behaviour of the thermal weight factor of modes with $\widetilde{\omega}$ close to 0 renders t.e.v.s infinite, causing rotating thermal states for scalar fields to be ill-defined everywhere in the space–time [2,3]. As discussed by [2,3], a resolution to these problems is to enclose the system inside a boundary located inside or on the SOL, restricting wavelengths such that $\widetilde{\omega}$ stays positive for all values of m.

3. Rotating fermions

In the Cartesian gauge [4], a natural frame for the metric (1) can be chosen to be:

$$e_{\hat{t}} = \partial_t - \Omega\partial_\varphi, \qquad e_{\hat{i}} = \partial_i. \tag{9}$$

In the following, hats shall be used to indicate tensor components with respect to the tetrad, i.e. $A^\mu = A^{\hat{\alpha}} e_{\hat{\alpha}}^\mu$. The Dirac equation for fermions of mass μ takes the form:

$$\left[\gamma^{\hat{t}}(H + \Omega M_z) - \boldsymbol{\gamma} \cdot \mathbf{P} - \mu\right]\psi(x) = 0, \tag{10}$$

where the gamma matrices are in the Dirac representation [5] and the covariant derivatives are given by:

$$iD_{\hat{t}} = H + \Omega M_z, \qquad -iD_{\hat{j}} = P_j. \tag{11}$$

The momentum operators P_j and angular momentum operator M_z are:

$$P_j = -i\partial_j, \qquad M_z = -i\partial_\varphi + \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \tag{12}$$

The Dirac equation (10) admits the following solutions:

$$U_{Ekm}^\lambda(x) = \frac{1}{\sqrt{8\pi^2}} e^{-i\widetilde{E}t + ikz} \begin{pmatrix} \sqrt{1 + \frac{\mu}{E}}\phi_{Ekm}^\lambda \\ \frac{2\lambda E}{|E|}\sqrt{1 - \frac{\mu}{E}}\phi_{Ekm}^\lambda \end{pmatrix}, \tag{13}$$

where the two-spinor ϕ_{Ekm}^λ is defined as:

$$\phi_{Ekm}^\lambda(\rho, \varphi) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + \frac{2\lambda k}{p}} e^{im\varphi} J_m(q\rho) \\ 2i\lambda\sqrt{1 - \frac{2\lambda k}{p}} e^{i(m+1)\varphi} J_{m+1}(q\rho) \end{pmatrix}, \tag{14}$$

where λ is the helicity [4,5], $p = \sqrt{q^2 + k^2}$ is the magnitude of the momentum and $E = \pm\sqrt{p^2 + \mu^2}$ controls the sign of the Minkowski energy of the mode. The eigenvalues of the Hamiltonian are $\widetilde{E} = E - \Omega(m + \frac{1}{2})$, representing, as in the scalar case, the energy seen by a co-rotating observer. The notations $j = (E_j, k_j, m_j, \lambda_j)$ and

$$\delta(j, j') = \delta_{\lambda_j \lambda_{j'}} \delta_{m_j m_{j'}} \delta(k_j - k_{j'}) \frac{\delta(E_j - E_{j'})}{|E_j|} \tag{15}$$

are useful to refer to modes and their norms. The latter can be computed using the Dirac inner product:

$$\langle\psi, \chi\rangle = \int d^3x \sqrt{-g}\, \psi^\dagger(x)\chi(x). \tag{16}$$

It can be shown that $\langle U_j, U_{j'}\rangle = \delta(j, j')$ for all possible labels j, j'. After choosing a suitable definition for particle modes (i.e. a range for the labels in j), the anti-particle modes can be constructed using charge conjugation [4,5]: $V_j = i\gamma^{\hat{2}} U_j^*$. Hence, V_j automatically inherits the same normalisation as U_j, namely: $\langle V_j, V_{j'}\rangle = \delta(j, j')$. Therefore there is no restriction on how the split into particle and anti-particle modes is performed, as long as the charge conjugation symmetry is preserved.

According to Vilenkin [2], the definition of particles for co-rotating observers should be the same as for inertial Minkowski observers, with the field operator written as:

$$\psi_V(x) = \sum_{\lambda_j=\pm\frac{1}{2}} \sum_{m_j=-\infty}^{\infty} \int_{\mu}^{\infty} E_j \, dE_j \int_{-p_j}^{p_j} dk_j \times \left[U_j(x) b_{j;V} + V_j(x) d_{j;V}^\dagger\right]. \tag{17}$$

Vilenkin's quantisation is equivalent to the one suggested by Letaw and Pfautsch [1] for the scalar field, yielding a vacuum state equivalent to the Minkowski vacuum. In contrast, Iyer [6] argues that the modes which represent particles for a co-rotating observer have positive frequency with respect to the co-rotating Hamiltonian, implying the following expression for the field operator:

$$\psi_I(x) = \sum_{\lambda_j=\pm\frac{1}{2}} \sum_{m_j=-\infty}^{\infty} \int_{\widetilde{E}_j>0, |E_j|>\mu}^{\infty} E_j \, dE_j \int_{-p_j}^{p_j} dk_j \times \left[U_j b_{j;I} + V_j(x) d_{j;I}^\dagger\right], \tag{18}$$

with the integral with respect to E_j running over both positive and negative values of E_j, as long as $\widetilde{E}_j > 0$ and $|E_j| > \mu$. Both quan-