Physics Letters B 734 (2014) 328–332

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Dynamics of monopole walls

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ARTICLE INFO

Article history:
Received 21 May 2014
Accepted 22 May 2014
Available online 28 May 2014
Editor: L. Alvarez-Gaumé

Keywords:
Monopole
Moduli-space
Monopole-wall

ABSTRACT

The moduli space of centred Bogomolny–Prasad–Sommerfield 2-monopole fields is a 4-dimensional manifold $\mathcal{M}$ with a natural metric, and the geodesics on $\mathcal{M}$ correspond to slow-motion monopole dynamics. The best-known case is that of monopoles on $\mathbb{R}^3$, where $\mathcal{M}$ is the Atiyah–Hitchin space. More recently, the case of monopoles periodic in one direction (monopole chains) was studied a few years ago. Our aim in this note is to investigate $\mathcal{M}$ for doubly-periodic fields, which may be visualized as monopole walls. We identify some of the geodesics on $\mathcal{M}$ as fixed-point sets of discrete symmetries, and interpret these in terms of monopole scattering and bound orbits, concentrating on novel features that arise as a consequence of the periodicity.

1. Introduction

The observation that the dynamics of Bogomolny–Prasad–Sommerfield (BPS) monopoles can be approximated as geodesics on the moduli space $\mathcal{M}$ of static solutions [1] has proved to be far-reaching. Not only does it reveal much about monopole dynamics, but the moduli spaces themselves are of considerable interest, for example in string theory. The best-known case is that of the centred 2-monopole system on $\mathbb{R}^3$, where $\mathcal{M}$ is a 4-dimensional asymptotically-locally-flat (ALF) space, namely the Atiyah–Hitchin manifold [2,3]. For monopoles periodic in one direction, in other words on $\mathbb{R}^2 \times S^1$, the asymptotic behaviour of the centred 2-monopole moduli space is different, and is called ALG [4]. In this case, the generalized Nahm transform has been used to describe some of the geodesics on the moduli space, and their interpretation in terms of periodic monopole dynamics [5,6].

This paper focuses on the doubly-periodic case, namely BPS monopoles on $T^2 \times \mathbb{R}$, also referred to as monopole walls [7,8]. An N-monopole field which is periodic in the x- and y-directions may be viewed as a set of N monopole walls, each extended in the xy-direction. Much is known about the general classification of the moduli spaces of such solutions, and their string-theoretic interpretation [8,9]. We shall restrict our attention here to the case of smooth 2-monopole fields with gauge group SU(2); the centred moduli space $\mathcal{M}$ is then a four-dimensional hyperkähler manifold with so-called ALH boundary behaviour [10]. The asymptotic form of its metric has recently been derived [11]. Our aim here is to identify some of the geodesics on $\mathcal{M}$ as fixed-point sets of discrete symmetries, and to interpret these in terms of monopole scattering, concentrating on novel features that arise as a consequence of the periodicity.

The system, therefore, consists of a smooth SU(2) gauge potential A_j on $T^2 \times \mathbb{R}$, plus a Higgs field Φ in the adjoint representation. The fields satisfy the Bogomolny equation $D_j\Phi = -B_j$, where $B_j = \frac{1}{2}\varepsilon_{jkl}F_{kl}$ is the SU(2) magnetic field. The coordinates are $x^j = (x, y, z)$, where x and y are periodic with period 1, and $z \in \mathbb{R}$. The boundary condition (see [7,8] for more detail) is $|\Phi|/|z| \to$ const as $z \to \pm\infty$. There are two topological charges $Q_\pm$, which are non-negative integers defined in terms of the winding number of Φ. More precisely, if $\Phi_c := \Phi|_{z=c}$, then $\hat{\Phi}_c := \Phi_c/|\Phi_c|$ is a map from T^2 to S^2, and we define $Q_\pm := \pm\deg\hat{\Phi}_{\pm c}$ for $c \gg 1$. The number of monopoles is $N = Q_+ + Q_-$, and we are interested in the case $N = 2$, so there are three possibilities, namely $(Q_-, Q_+) = (1, 1)$, $(0, 2)$ or $(2, 0)$. In fact, the corresponding moduli spaces are isometric [9]. In what follows, we shall concentrate on the $(1, 1)$ wall, namely $Q_- = Q_+ = 1$.

2. Parameters and moduli of the (1, 1) wall

We begin by reviewing the parameters, the moduli, the energy, and the spectral data of the $(1, 1)$ wall, using the same conventions and notation as in [8]. There exists a (non-periodic) gauge such that the boundary behaviour of the fields is

$$\Phi \sim 2\pi\mathrm{i}(z + M_\pm)\sigma_3, \quad A_j \to \pi\mathrm{i}(y - 2p_\pm, -x - 2q_\pm, 0)\sigma_3 \qquad (1)$$

as $z \to \pm\infty$. The six real constants $(M_\pm, p_\pm, q_\pm)$ are the boundary-value parameters, with $M_\pm \in \mathbb{R}$ and $p_\pm, q_\pm \in (-\frac{1}{2}, \frac{1}{2}]$. Fixing the

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http://dx.doi.org/10.1016/j.physletb.2014.05.072
0370-2693/

centre-of-mass of the system amounts to fixing (M_-, p_-, q_-) in terms of the other three parameters (M_+, p_+, q_+). Henceforth, we fix the centre-of-mass to be at the point $(x, y, z) = (\frac{1}{2}, \frac{1}{2}, 0)$, and the field is then invariant (up to a gauge transformation) under the map $(x, y, z) \mapsto (1-x, 1-y, -z)$ plus $\Phi \mapsto -\Phi$. In effect, the system as a whole has infinite mass, and only the relative separation and phase of the two monopoles appear in the moduli space; the space of fields with fixed $(M_\pm, p_\pm, q_\pm)$, modulo gauge transformations, is our four-dimensional moduli space $\mathcal{M}$.

The energy density is $\mathcal{E} = |D\Phi|^2 + |B|^2$, and $\mathcal{E} \to 8\pi^2$ as $z \to \pm\infty$. The total energy, *i.e.* $\mathcal{E}$ integrated over $T^2 \times \mathbb{R}$, is consequently infinite. But the cut-off energy

$$E_L = \int_{-L}^{L} dz \int \left(|D\Phi|^2 + |B|^2\right) dx dy \tag{2}$$

is finite, and if $L \gg -M_+$ it equals the Bogomolny bound [7]

$$E_L = 16\pi^2 (L + M_+). \tag{3}$$

Spectral data for this system may be defined as follows [8]. Put

$$W_x = \operatorname{tr} \mathcal{P} \exp \int_0^1 (-A_x - \mathrm{i}\Phi) dx,$$

$$W_y = \operatorname{tr} \mathcal{P} \exp \int_0^1 (-A_y - \mathrm{i}\Phi) dy.$$

Then W_x and W_y have the form

$$W_x = W_x(s) = \left(s + s^{-1}\right) \exp\left[2\pi (M_+ + \mathrm{i}p_+)\right] + 2D_x, \tag{4}$$

$$W_y = W_y(\tilde{s}) = \left(\tilde{s} + \tilde{s}^{-1}\right) \exp\left[2\pi (M_+ + \mathrm{i}q_+)\right] + 2D_y, \tag{5}$$

where $s = \exp[2\pi(z - \mathrm{i}y)]$ and $\tilde{s} = \exp[2\pi(z + \mathrm{i}x)]$, and where D_x, D_y are complex constants. The real and imaginary parts of D_x and D_y are moduli; but they are not independent, so do not provide all the moduli.

The Nahm transform maps walls to walls, although in general the gauge group, the topological charges, and the number of Dirac singularities change [8,9]. In our case, however, these properties do not change: the Nahm transform of a smooth SU(2) wall of charge (1, 1) is again of that type. The action of a Nahm transform on the parameters and the moduli is as follows:

$$(M_+, p_+, q_+) \mapsto (-M_+, -p_+, -q_+), \tag{6}$$

$$D_x \mapsto -D_x \exp\left[-2\pi (M_+ + \mathrm{i}p_+)\right], \tag{7}$$

$$D_y \mapsto -D_y \exp\left[-2\pi (M_+ + \mathrm{i}q_+)\right]. \tag{8}$$

These expressions follow from the fact that the x-spectral curve, given by $t^2 - tW_x(s) + 1 = 0$, is invariant under the Nahm transform, which acts by interchanging the variables t and s; and similarly for the y-spectral curve [8].

3. The asymptotic region of $\mathcal{M}$

In order to understand the role played by the parameters and the moduli, let us first look at the asymptotic region of moduli space $\mathcal{M}$, which consists of those fields for which $|\Phi|_{z=0} \gg 1$. It follows from this condition that D_x and D_y have the approximate form

$$D_x \approx \cosh\left[2\pi (M + \mathrm{i}p)\right], \quad D_y \approx \cosh\left[2\pi (M + \mathrm{i}q)\right], \tag{9}$$

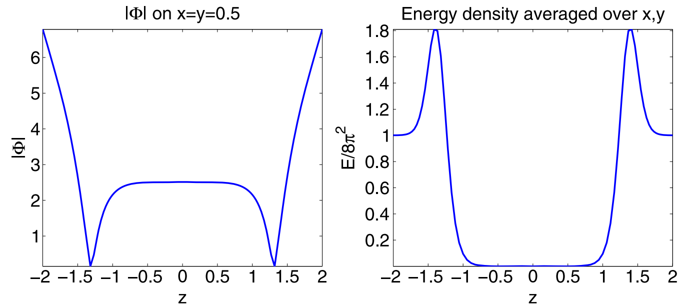

Fig. 1. Higgs field and energy density of a well-separated two-wall solution.

with $M \gg \max\{1, M_+\}$. Three of the four asymptotic moduli are M and $p, q \in (-\frac{1}{2}, \frac{1}{2}]$. The walls are located at values of z for which $W_x(s)$ has zeros, and we see from (4) that this occurs for $z = z_\pm = \pm(M - M_+)$; so we have two well-separated walls. Note that $|D_x| \approx |D_y|$ up to exponentially small corrections, so we could equally well have used the zeros of $W_y(\tilde{s})$ to define the wall locations; but this is only true asymptotically, and not in the core region of $\mathcal{M}$. Each wall has a monopole embedded in it, the monopole locations $\mathbf{R}_\pm = (x_\pm, y_\pm, z_\pm)$ being defined to be where $W_x(s) = 0 = W_y(\tilde{s})$. Numerical solutions indicate that this is where Φ is zero, and also where the energy density is peaked. It follows from (4, 5) that the location of the $z > 0$ monopole is $\mathbf{R}_+ = (\frac{1}{2} + q - q_+, \frac{1}{2} - p + p_+, M - M_+)$.

The energy density is approximately zero for $z_- < z < z_+$ (between the two walls), and tends to $8\pi^2$ as $z \to \pm\infty$. See Fig. 1, which depicts a solution with $M_+ = -0.92$ and $D_x = D_y = 6.21$; this solution was obtained numerically by minimizing the functional (2). The left-hand plot is of $|\Phi|$ on the line $x = y = \frac{1}{2}$, where the monopoles are located. The right-hand plot is of the normalized, xy-averaged energy density $(8\pi^2)^{-1} \int \mathcal{E} dx dy$, as a function of z. Between the walls, the function $|\Phi|$ is approximately constant; in fact $|\Phi| \approx 2\pi M$.

In view of the shape of the energy density, one might have expected that E_L could be reduced by moving the walls further apart, *i.e.* by increasing M: it looks like an increase δM in M would give $\delta E_L = -16\pi^2 \delta M$, as the central region (where $\mathcal{E}$ is zero) increases in size. But in fact as M increases and the walls move apart, the energy contained in each monopole increases by $8\pi^2 \delta M$. This is because each monopole resembles an $\mathbb{R}^3$ monopole with $|\Phi|_\infty = 2\pi M$ and therefore energy $8\pi^2 M$. So the total energy E_L is independent of M, as it must be from (3). Note, however, that stability involves fixing the value of the parameter M_+, and reducing M_+ really does lower the energy. This is analogous to having to fix the boundary value of $|\Phi|$ in the $\mathbb{R}^3$ case.

Furthermore, the size of each monopole core is proportional to M^{-1}, and therefore one may think of them as small SU(2) monopoles embedded in an ambient U(1) field. So the asymptotic moduli are analogous to those of the $\mathbb{R}^3$ case: three moduli (M, p, q) determine the relative location of the two monopoles, and the fourth is a relative phase $\omega \in (-\pi, \pi]$ between them. The asymptotic metric, in our coordinates (M, p, q, ω), takes the hyperkähler form [11]

$$ds^2 = \pi W \left(dM^2 + dp^2 + dq^2\right) + \pi W^{-1}\left[d\omega - 8\pi(q dp - p dq)\right]^2, \tag{10}$$

where $W = W(M) = 8\pi(2M - M_+)$. Here, for simplicity, we have set $p_+ = q_+ = 0$. Note from (10) that $R = M^{3/2}$ is an affine parameter on asymptotic 'radial' geodesics p, q, ω constant. The volume Vol_R of a ball of radius R scales like $\mathrm{Vol}_R \sim R^{4/3}$, and so $\mathcal{M}$ is of ALH type [10].