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The general property of dynamical quintessence field

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ABSTRACT

We discuss the general dynamical behaviors of quintessence field, in particular, the general conditions for tracking and thawing solutions are discussed. We explain what the tracking solutions mean and in what sense the results depend on the initial conditions. Based on the definition of tracking solution, we give a simple explanation on the existence of a general relation between w_ϕ and Ω_ϕ which is independent of the initial conditions for the tracking solution. A more general tracker theorem which requires large initial values of the roll parameter is then proposed. To get thawing solutions, the initial value of the roll parameter needs to be small. The power-law and pseudo-Nambu Goldstone boson potentials are used to discuss the tracking and thawing solutions. A more general w_ϕ–Ω_ϕ relation is derived for the thawing solutions. Based on the asymptotical behavior of the w_ϕ–Ω_ϕ relation, the flow parameter is used to give an upper limit on w'_ϕ for the thawing solutions. If we use the observational constraint $w_{\phi 0} < -0.8$ and $0.2 < \Omega_{m0} < 0.4$, then we require $n \lesssim 1$ for the inverse power-law potential $V(\phi) = V_0(\phi/m_{pl})^{-n}$ with tracking solutions and the initial value of the roll parameter $|\lambda_i| < 1.3$ for the potentials with the thawing solutions.

1. Introduction

The recent cosmic acceleration observed by type Ia supernova data [1] was usually explained by introducing a dynamical scalar field called quintessence [2–4]. More general dynamical scalar field models such as phantom [5], quintom [6], tachyon [7] and k-essence [8] were also proposed. For a recent review of dark energy, please see Ref. [9].

For a dynamical scalar field ϕ with the potential $V(\phi)$ in the flat Friedmann–Lemaître–Robertson–Walker universe with the metric $ds^2 = -dt^2 + a^2(t)(dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2)$, its energy density and pressure are $\rho_\phi = \dot\phi^2/2 + V(\phi)$ and $p_\phi = \dot\phi^2/2 - V(\phi)$, where $\dot\phi = d\phi/dt$. The scalar field rolls down a very shallow potential while its equation of state $w_\phi = p_\phi/\rho_\phi$ approaches -1 and it starts to dominate the Universe recently. Because the scalar field catches up the background only recently and the current value of its equation of state parameter is around -1, w_ϕ does not change too much in the redshift range $0 \leqslant z < 1$ for most scalar fields, so the time variation of w_ϕ is bounded for the thawing and freezing models [10–18]. In general, the evolution of scalar field depends on the initial conditions. However, the attractor solutions and the tracking solutions are independent of the initial conditions [19–36]. In particular, the tracker field ϕ tracks below the background density for most of the history of the Universe until it starts to dominate recently for a wide range of initial conditions, and there exists a relation between w_ϕ and the fractional energy density $\Omega_\phi = 8\pi G\rho_\phi/(3H^2)$ today, where the Hubble parameter $H(t) = \dot a/a$. There also exists a general w_ϕ–Ω_ϕ relation for the thawing solutions which is well approximated by some analytical expressions [37–47]. In this Letter, we will discuss the general dynamics such as the w_ϕ–Ω_ϕ relation and the bound on $w'_\phi = dw_\phi/d\ln a$ of the tracking and thawing fields. We use the power-law potential and the pseudo-Nambu Goldstone boson (PNGB) potential [48–52] as examples to illustrate the general dynamical behaviors of tracking and thawing fields.

If the Universe is filled with the quintessence field and the background matter with the equation of state $w_b = [(1/3)a_{eq}/a]/[1 + a_{eq}/a]$, where $a_{eq} = 1/3403$ [53] is the scale factor $a(t)$ at the matter-radiation equality, then in terms of the dimensionless variables,

$$x = \frac{\phi'}{\sqrt{6}} = \frac{1}{\sqrt{6}}\frac{d\phi}{d\ln a}, \qquad y = \sqrt{\frac{V}{3H^2}},$$
$$\lambda = -\frac{V_{,\phi}}{V} = -\frac{1}{V}\frac{dV}{d\phi}, \qquad \Gamma = \frac{V V_{,\phi\phi}}{V_{,\phi}^2}, \tag{1}$$

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the cosmological equations are

$$x' = \sqrt{\frac{3}{2}}\lambda y^2 + \frac{3}{2}x\left(x^2 - y^2 - 1\right) + \frac{3}{2}w_b x\left(1 - x^2 - y^2\right), \tag{2}$$

$$y' = -\sqrt{\frac{3}{2}}\lambda xy + \frac{3}{2}y\left(1 + x^2 - y^2\right) + \frac{3}{2}w_b y\left(1 - x^2 - y^2\right), \tag{3}$$

$$\lambda' = -\sqrt{6}\lambda^2(\Gamma - 1)x. \tag{4}$$

The fractional energy density and the equation of state of the scalar field are

$$\Omega_\phi = x^2 + y^2, \qquad w_\phi = \frac{x^2 - y^2}{x^2 + y^2}. \tag{5}$$

Using the fractional energy density Ω_ϕ and the parameter $\gamma = 1 + w$, Eqs. (2)–(4) become

$$\Omega_\phi' = 3(\gamma_b - \gamma_\phi)\Omega_\phi(1 - \Omega_\phi), \tag{6}$$

$$\gamma_\phi' = (2 - \gamma_\phi)\left(-3\gamma_\phi + |\lambda|\sqrt{3\gamma_\phi\Omega_\phi}\right), \tag{7}$$

$$\lambda' = -\sqrt{3\gamma_\phi\Omega_\phi}\lambda|\lambda|(\Gamma - 1). \tag{8}$$

From Eq. (7), we get a lower limit $w_\phi' \geqslant -3(1 + w_\phi)(1 - w_\phi)$. If the tracker parameter Γ can be expressed as a function of the roll parameter λ, then the above system (6)–(8) becomes an autonomous system. In this case, we have additional critical point $\Omega_{\phi c} = 1$ and $\gamma_c = 0$ which is absent in the system (2)–(4), where the subscript c means the critical point. At the point $x = 0$ and $y = 1$, the transformation from (x, y) to (Ω_ϕ, γ) is singular, so we get different critical points. Only when $\lambda_c = 0$, the point $x = 0$ and $y = 1$ is the critical point of the system (2)–(4). For the exponential potential, the point $x = 0$ and $y = 1$ is not a critical point for the system (2)–(4) and the critical point $\Omega_{\phi c} = 1$ and $\gamma_c = 0$ for the system (6)–(8) is not a stable point.

If we use the flow parameter $F = \gamma_\phi/(\Omega_\phi\lambda^2)$ [16], then Eq. (7) can be written as

$$\gamma_\phi' = 3\gamma_\phi(2 - \gamma_\phi)\left(-1 + 1/\sqrt{3F}\right). \tag{9}$$

To understand the general dynamics of the quintessence, it is useful to use the function $\beta = \ddot{\phi}/(3H\dot{\phi})$ [16,44],

$$\begin{aligned}\beta &= -1 + \frac{1 - w_\phi}{\sqrt{12F}} \\ &= \frac{1}{2}\left[\Omega_\phi\gamma_\phi + (1 - \Omega_\phi)\gamma_b\right] - \frac{\beta'}{3(1 + \beta)} - \frac{V_{,\phi\phi}}{9(1 + \beta)H^2}.\end{aligned} \tag{10}$$

For the thawing solution, the quintessence field rolls down the potential very slowly, $V_{,\phi\phi} \approx 0$ and β is almost a constant, so $\beta \approx \gamma_b/2$ at early time when $\Omega_\phi \approx 0$ and $w_\phi \approx -1$ [44].

2. Tracker solution

The energy density of the tracker field ϕ tracks below the background density for most of the history of the Universe, it starts to dominate the energy density only recently and then drives the cosmic acceleration. The tracker fields have attractor-like solutions in the sense that they rapidly converge to a common cosmic evolutionary track of $\rho_\phi(t)$ and $w_\phi(t)$ for a very wide range of initial conditions, so the tracking solutions are extremely insensitive to the initial conditions [4,22]. Furthermore, an important relation between w_ϕ and Ω_ϕ today was found for the tracker field. When the tracker field enters the tracking solution, it satisfies the tracker condition [22]

$$\gamma_\phi = 1 + w_\phi = \frac{1}{3}\lambda^2\Omega_\phi, \tag{11}$$

thus this condition is the initial condition of tracking solution. In other words, the initial condition for the tracking solution reads $F = 1/3$. From Eq. (7), we see that $\gamma_\phi' = 0$ when the tracker condition is satisfied, so it is possible that w_ϕ stops varying. On the other hand, the quintessence field satisfies the tracker equation [22,28]

$$\begin{aligned}\Gamma - 1 = {}& \frac{3(w_b - w_\phi)(1 - \Omega_\phi)}{(1 + w_\phi)(6 + \tilde{x}')} - \frac{(1 - w_\phi)\tilde{x}'}{2(1 + w_\phi)(6 + \tilde{x}')} \\ & - \frac{2\tilde{x}''}{(1 + w_\phi)(6 + \tilde{x}')^2},\end{aligned} \tag{12}$$

where $\tilde{x} = \ln[(1 + w_\phi)/(1 - w_\phi)]$ and $\tilde{x}' = d\ln\tilde{x}/d\ln a$. For the tracking solution, w_ϕ is nearly constant, so $\tilde{x}' \approx \tilde{x}'' \approx 0$, and we get

$$w_\phi \approx \frac{w_b(1 - \Omega_\phi) - 2(\Gamma - 1)}{2\Gamma - 1 - \Omega_\phi} < w_b \quad (\Gamma > 1). \tag{13}$$

If $\Omega_\phi \approx 0$ when the tracker condition (11) is satisfied, then

$$w_\phi = w_\phi^{trk} = \frac{w_b - 2(\Gamma - 1)}{2\Gamma - 1}, \tag{14}$$

and $\beta = -\gamma_b/2(2\Gamma - 1)$ are approximately constants if the tracker parameter Γ is nearly constant, $\Omega_\phi \propto a^{6\gamma_b(\Gamma-1)/(2\Gamma-1)}$ increases with time and $\lambda^2 \approx 3(1 + w_\phi^{trk})/\Omega_\phi$ decreases with time. For the tracker field, $V_{,\phi\phi}/H^2$ is not negligible, so $\beta \neq \gamma_b/2$. In fact, $V_{,\phi\phi}/H^2$ is a constant for the exponential potential when the attractor is reached.

If Ω_ϕ is not small or the tracker parameter changes rapidly when the tracker condition (11) is satisfied, then w_ϕ won't keep to be a time independent constant and it decreases with time while Ω_ϕ increases to 1, so the scalar field does not track the background and Eq. (14) does not hold when the tracker condition (11) is satisfied, but the scalar field has the freezing behavior with $w_\phi \to -1$ asymptotically. Therefore, both the tracker condition (11) and Eq. (14) will be violated when Ω_ϕ is not negligible or Γ changes rapidly, and w_ϕ keeps decreasing.

For the tracker field, the tracking solution at late times has the property that $\gamma_\phi \to 0$ and $\Omega_\phi \to 1$, so γ_ϕ should decrease with time while Ω_ϕ increases with time. When γ_ϕ reaches the background value γ_b, and λ^2 decreases to the value $\lambda^2 = 3\gamma_\phi/\Omega_\phi$, then we reach the tracker condition. After that, γ_ϕ decreases toward to zero and Ω_ϕ increases toward 1. From Eq. (7), we know that we should keep $|\lambda| < \sqrt{3\gamma_\phi/\Omega_\phi}$ in order that $\gamma_\phi' < 0$, therefore $|\lambda|$ should decrease with time and $\lambda \to 0$ when $\gamma_\phi \to 0$. For any quintessence field rolling down its potential, $|\lambda|$ does not increase with time is equivalent to $\Gamma \geqslant 1$ as easily seen from Eq. (8).

For the exponential potential, λ is a constant and $\Gamma = 1$. If λ is small, then eventually γ_ϕ will decrease to be less than γ_b, and Ω_ϕ will quickly increase to be 1. In particular, if $\lambda^2 < 3\gamma_b$, then the system will reach the attractor solution with $\Omega_\phi = 1$ and $\gamma_\phi = \lambda^2/3$. If λ is big, i.e., $\lambda^2 \geqslant 3\gamma_b$, then the attractor solution is $\gamma_\phi = \gamma_b = \lambda^2\Omega_\phi/3$. Since the above attractors satisfy the tracker condition (11), so both of them are also tracking solutions. In Fig. 1, we show the phase diagram for the exponential potential with $\lambda = 2.1$. The original tracking solution found in [22] is independent of the value of λ which is in contradiction with the results for the exponential potential. The contradiction was then resolved in [28] by deriving the correct tracker equation (12).

With the dynamical Eqs. (6)–(8), we can understand the general dynamical evolution of the tracker field as follows: (a) Initially if $\Omega_{\phi i}$ is not too small or λ_i is large enough so that $\lambda_i^2 > 3\gamma_{\phi i}/\Omega_{\phi i}$, where the subscript i means the initial value, then γ_ϕ will increase toward 2 independent of the initial value of w_ϕ. Once $\gamma_\phi > \gamma_b$,