Physics Letters B 731 (2014) 327–330

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

On the κ-Dirac oscillator revisited

F.M. Andrade [a,*], E.O. Silva [b], M.M. Ferreira Jr. [b], E.C. Rodrigues [b]

[a] Departamento de Matemática e Estatística, Universidade Estadual de Ponta Grossa, 84030-900 Ponta Grossa, PR, Brazil
[b] Departamento de Física, Universidade Federal do Maranhão, Campus Universitário do Bacanga, 65085-580 São Luís, MA, Brazil

ARTICLE INFO

Article history:
Received 11 December 2013
Received in revised form 13 February 2014
Accepted 25 February 2014
Available online 5 March 2014
Editor: M. Cvetič

Keywords:
κ-Poincaré–Hopf algebra
Dirac oscillator

ABSTRACT

This Letter is based on the κ-Dirac equation, derived from the κ-Poincaré–Hopf algebra. It is shown that the κ-Dirac equation preserves parity while breaks charge conjugation and time reversal symmetries. Introducing the Dirac oscillator prescription, $\mathbf{p} \to \mathbf{p} - im\omega\beta\mathbf{r}$, in the κ-Dirac equation, one obtains the κ-Dirac oscillator. Using a decomposition in terms of spin angular functions, one achieves the deformed radial equations, with the associated deformed energy eigenvalues and eigenfunctions. The deformation parameter breaks the infinite degeneracy of the Dirac oscillator. In the case where $\varepsilon = 0$, one recovers the energy eigenvalues and eigenfunctions of the Dirac oscillator.

1. Introduction

In 1989, in a seminal paper by Moshinsky and Szczepaniak [1] the basic idea of a relativistic quantum mechanical oscillator, called Dirac oscillator, was proposed. Such oscillator behaves as an harmonic oscillator with a strong spin-orbit coupling in the non-relativistic limit. Since the time of its proposal it has been the object of considerable attention in various branches of theoretical physics. For instance, it appears in mathematical physics [2–11], nuclear physics [12–14], quantum optics [15–18], supersymmetry [19–21], and noncommutativity [22–25]. Recently, the first experimental realization of the Dirac oscillator was realized by J.A. Franco-Villafañe et al. [26], which should draw even more attention for such system. Moreover, C. Quibay et al. proposed that the Dirac oscillator can describe some electronic properties of monolayer and bylayer graphene [27] and show the existence of a quantum phase transition in this system [28].

The Dirac oscillator has also been discussed in connection with the theory of quantum deformations [29]. Some of these deformations are based on the κ-deformed Poincaré–Hopf algebra, with κ being a masslike fundamental deformation parameter, introduced in Refs. [30,31] and further discussed in Refs. [32–35]. The κ-deformed algebra is defined by the following commutation relations:

$$[p_\nu, p_\mu] = 0, \tag{1a}$$

$$[M_i, p_\mu] = (1-\delta_{0\mu}) i\epsilon_{ijk} p_k, \tag{1b}$$

$$[L_i, p_\mu] = i[p_i]^{\delta_{0\mu}} \big[\delta_{ij}\varepsilon^{-1}\sinh(\varepsilon p_0)\big]^{1-\delta_{0\mu}}, \tag{1c}$$

$$[M_i, M_j] = i\epsilon_{ijk} M_k, \qquad [M_i, L_j] = i\epsilon_{ijk} L_k, \tag{1d}$$

$$[L_i, L_j] = -i\epsilon_{ijk}\left[M_k\cosh(\varepsilon p_0) - \frac{\varepsilon^2}{4} p_k p_l M_l\right], \tag{1e}$$

where ε is defined by

$$\varepsilon = \kappa^{-1} = \lim_{R\to\infty}(R\ln q), \tag{2}$$

with R being the de Sitter curvature, q is a real deformation parameter, and $p_\mu = (p_0, \mathbf{p})$ is the κ-deformed generator for energy and momenta. Also, the M_i, L_i represent the spatial rotations and deformed boosts generators, respectively. The coalgebra and antipode for the κ-deformed Poincaré–Hopf algebra was established in Ref. [36].

Several investigations have been developed in the latest years in the context of this theoretical framework on space-like κ-deformed Minkowski spacetime. The interest in this issue also appears in field theories [37–40], quantum electrodynamics [41–43], realizations in terms of commutative coordinates and derivatives [44–47], relativistic quantum systems [48–52], doubly special relativity [53], noncommutative black holes [54] and the construction of scalar theory [55].

The aim of this Letter is to suitably describe the κ-Dirac oscillator making use of the κ-Poincaré–Hopf algebra, tracing a comparison with the results of Ref. [29], where it was argued that usual approach for introducing the Dirac oscillator, $\mathbf{p} \to \mathbf{p} - im\omega\beta\mathbf{r}$, in the κ-Dirac equation [32,33], has not led to the Dirac oscillator

* Corresponding author.
E-mail addresses: fmandrade@uepg.br (F.M. Andrade), edilbertoos@pq.cnpq.br (E.O. Silva), manojr.ufma@gmail.com (M.M. Ferreira), ednilson.fisica@gmail.com (E.C. Rodrigues).

http://dx.doi.org/10.1016/j.physletb.2014.02.054
0370-2693/

spectrum in the limit $\varepsilon \to 0$. This result, however, contradicts the well-known fact that the κ-Dirac equation recovers the standard Dirac equation in this limit. In this context, this Letter reassessed the κ-Dirac oscillator problem yielding a modified oscillator spectrum that indeed regains the Dirac oscillator behavior in the limit $\varepsilon \to 0$.

The plan of our Letter is the following. In Section 2 we introduce the κ-Dirac analyzing its behavior under $\mathcal{C}$, $\mathcal{P}$, $\mathcal{T}$ (discrete) symmetries. In Section 3 the oscillator prescription is implemented in order to study the physical implications of the κ-deformation in the Dirac oscillator problem. Using a decomposition in terms of spin angular functions, we write the relevant radial equation to study the dynamics of the system. Section 4 is devoted to the calculation the energy eigenvalues and eigenfunctions of the κ-Dirac oscillator and to the discussion of the results. A brief conclusion in outlined in Section 5.

2. κ-Dirac equation and discrete symmetries

In this section, we present κ-Dirac equation, invariant under the κ-Poincaré quantum algebra [32], considering $O(\varepsilon)$ [33]:

$$\left\{(\gamma_0 p_0 - \gamma_i p_i) + \frac{\varepsilon}{2}\left[\gamma_0\left(p_0^2 - p_i p_i\right) - m p_0\right]\right\}\psi = m\psi, \tag{3}$$

which recovers the standard Dirac equation in the limit $\varepsilon \to 0$.

An initial discussion refers to the behavior of this deformed equation under $\mathcal{C}$, $\mathcal{P}$, $\mathcal{T}$ (discrete) symmetries. Concerning the parity operator ($\mathcal{P}$), in the context of the Dirac equation, $\mathcal{P} = i\gamma^0$, with $\mathcal{P}\gamma^\mu\mathcal{P}^{-1} = \gamma_\mu$ and $\psi_P = \mathcal{P}\psi$ being the parity-transformed spinor. Applying $\mathcal{P}$ on the Dirac deformed equation, we attain

$$\left\{(\gamma_0 p_0 - \gamma_i p_i) + \frac{\varepsilon}{2}\left[\gamma_0\left(p_0^2 - p_i p_i\right) - m p_0\right]\right\}\psi_{\mathcal{P}} = m\psi_{\mathcal{P}}, \tag{4}$$

concluding that it is invariant under $\mathcal{P}$ action.

We can now verify that this equation is not invariant under charge conjugation ($\mathcal{C}$) and time reversal ($\mathcal{T}$). As for the $\mathcal{C}$ operation, the charge-conjugated spinor is $\psi_{\mathcal{C}} = U_{\mathcal{C}}\psi^* = \mathcal{C}\gamma^0\psi^*$, with $\mathcal{C} = i\gamma^2\gamma^0$ being the charge conjugation operator, and $U_{\mathcal{C}}\gamma^{\mu *}U_{\mathcal{C}}^{-1} = -\gamma^\mu$. On the other hand, the time reversal operator is, $\mathcal{T} = i\gamma^1\gamma^3$, so that $\psi_{\mathcal{T}}(x,t') = \mathcal{T}\psi^*(x,t')$, and $\mathcal{T}\gamma^{\mu *}\mathcal{T}^{-1} = (\gamma^0, -\gamma^i)$. Applying $U_{\mathcal{C}}$ and $\mathcal{T}$ on the complex conjugate of Eq. (3), we achieve:

$$\left\{(\gamma_0 p_0 - \gamma_i p_i) + \frac{\varepsilon}{2}\left[-\gamma_0\left(p_0^2 - p_i p_i\right) - m p_0\right]\right\}\psi_{\mathcal{C}} = m\psi_{\mathcal{C}}, \tag{5}$$

$$\left\{(\gamma_0 p_0 - \gamma_i p_i) + \frac{\varepsilon}{2}\left[(\gamma_0)\left(p_0^2 - p_i p_i\right) + m p_0\right]\right\}\psi_{\mathcal{T}} = m\psi_{\mathcal{T}}. \tag{6}$$

Theses equations differ from Eq. (3), revealing that the $\mathcal{C}$ and $\mathcal{T}$ are not symmetries of this system. As a consequence, particle and anti-particle eigenenergies should become different. Further, note that under $\mathcal{CT}$ or $\mathcal{CPT}$ operations the original equation is modified as

$$\left\{(\gamma_0 p_0 - \gamma_i p_i) - \frac{\varepsilon}{2}\left[\gamma_0\left(p_0^2 - p_i p_i\right) - m p_0\right]\right\}\psi' = m\psi', \tag{7}$$

where $\psi' = \psi_{\mathcal{CT}}$ or $\psi' = \psi_{\mathcal{CPT}}$, showing that this equation is not invariant under $\mathcal{CT}$ or $\mathcal{CPT}$ operations, once the parameter ε is always positive.

3. κ-Dirac oscillator equation

Now, we derive the equation that governs the dynamics of the Dirac oscillator in the context of Eq. (3). The Dirac oscillator stems from the prescription [1]

$$p_0 \to p_0 = H_0, \tag{8a}$$

$$\mathbf{p} \to \mathbf{p} - im\omega\beta\mathbf{r}, \tag{8b}$$

where $\mathbf{r}$ is the position vector, m is the mass of particle and ω the frequency of the oscillator. The κ-Dirac oscillator can be obtained by substituting Eq. (8) into Eq. (3). The result is

$$H\psi = E\psi, \tag{9}$$

with

$$H = H_0 - \frac{\varepsilon}{2}\left[p_0^2 - (\mathbf{p} - im\omega\beta\mathbf{r})(\mathbf{p} - im\omega\beta\mathbf{r}) - \beta m p_0\right], \tag{10}$$

where H_0 represents the undeformed part of the Dirac operator

$$H_0 = \boldsymbol{\alpha}\cdot(\mathbf{p} - im\omega\beta\mathbf{r}) + \beta m. \tag{11}$$

At this point it is important trace a comparison with the results of Ref. [29], in which it is argued that the prescription of Eq. (8), yielding the κ-deformed Hamiltonian of Eq. (10), does not lead to an oscillator-like spectrum even when $\varepsilon \to 0$. This result, however, is not correct, as properly shown in Section 4. Furthermore, another deformed wave equation is introduced without any kind of proof (see Eq. (15) in [29]). Here, instead of postulating a deformed wave equation, we follow a pragmatic approach obtaining the κ-Dirac oscillator equation (10) from basic principles.

In the four-dimensional representation, the matrices $\boldsymbol{\gamma}$ and $\boldsymbol{\alpha}$ are given by

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \boldsymbol{\gamma} = \beta\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \tag{12}$$

and obey the anticommutation relations and the square identity,

$$\{\alpha_i, \alpha_j\} = 0, \quad i \neq j,$$

$$\{\alpha_i, \beta\} = 0,$$

$$\alpha_i^2 = \beta^2 = I.$$

In the representation (12), ψ may be written as a bispinor $\psi = (\psi_1, \psi_2)^T$ in terms of two-component spinors ψ_1 and ψ_2. Thus, Eq. (9) leads to

$$\left(1 + \frac{m\varepsilon}{2}\right)(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}^+)\psi_2 = (E - m)\psi_1 + \varepsilon\left[im\omega(\mathbf{r}\cdot\mathbf{p}) + m\omega(\boldsymbol{\sigma}\cdot\mathbf{L}) + m^2\omega^2 r^2\right]\psi_1, \tag{13}$$

$$\left(1 - \frac{m\varepsilon}{2}\right)(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}^-)\psi_1 = (E + m)\psi_2 - \varepsilon\left[im\omega(\mathbf{r}\cdot\mathbf{p}) + m\omega(\boldsymbol{\sigma}\cdot\mathbf{L}) - m^2\omega^2 r^2\right]\psi_2, \tag{14}$$

where

$$\boldsymbol{\pi}^\pm = \mathbf{p} \pm im\omega\mathbf{r}. \tag{15}$$

Since we are interested in studying the κ-Dirac oscillator in a three-dimensional spacetime, Eqs. (13) and (14) above may be solved in spherical coordinates. First, using the property

$$\boldsymbol{\sigma}\cdot\mathbf{p} = (\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\left(\hat{\mathbf{r}}\cdot\mathbf{p} + i\frac{\boldsymbol{\sigma}\cdot\mathbf{L}}{r}\right), \tag{16}$$

with $\boldsymbol{\sigma}\cdot\mathbf{r} = r\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}$, we rewrite the quantity $\boldsymbol{\sigma}\cdot\boldsymbol{\pi}^\pm$ as

$$\boldsymbol{\sigma}\cdot\boldsymbol{\pi}^\pm = (\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\left(\hat{\mathbf{r}}\cdot\mathbf{p} + i\frac{\hat{\mathbf{K}} - 1}{r} \pm im\omega r\right), \tag{17}$$