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Iterative method to compute the Fermat points and Fermat distances of multiquarks

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ABSTRACT

The multiquark confining potential is proportional to the total distance of the fundamental strings linking the quarks and antiquarks. We address the computation of the total string distance and of the Fermat points where the different strings meet. For a meson the distance is trivially the quark–antiquark distance. For a baryon the problem was solved geometrically from the onset by Fermat and by Torricelli, it can be determined just with a rule and a compass, and we briefly review it. However we also show that for tetraquarks, pentaquarks, hexaquarks, etc., the geometrical solution is much more complicated. Here we provide an iterative method, converging fast to the correct Fermat points and the total distances, relevant for the multiquark potentials.

1. Introduction

Fermat proposed to Torricelli the problem of finding the point in a triangle minimizing the sum of the distances to the three respective vertices. This first Fermat point or Torricelli point [1–6], is the isogonic point, since in a sufficiently acute triangle the angle formed by the segments connecting any two vertices with it is 120 degrees.

Lately this problem became relevant for quark physics because the multiquark confining potential is proportional to the total distance of the fundamental strings linking the quarks and antiquarks. Recently the interest on multiquarks and other exotic hadrons has been increasing in the literature. The most recent focus is in plausible tetraquarks with some heavy quarks in the family of X, Y and Z mesons [7–26]. We may expect the experimental collaborations BESII/BEPC, PANDA, GLUEX and JPARK, to find evidence for new multiquarks in the future be discovered.

Here we address the case where we have a single multiquark linked by topologically connected confining strings, and not many free or molecular mesons and baryons, where the confining potential would be different. The three-body star-like potential has already been used long ago in baryons [26], however for many years there was a debate in the lattice QCD community on the two-body versus three body nature of the confining potential for baryons. Recently, the study of flux tubes in lattice QCD for baryons (triquarks) by Takahashi et al. [27] confirmed the three-body star-like confining potential. Very recently, the Wilson loop technique was applied to tetraquarks by Okiharu et al. and Alexandrou et al. [28,29] and to pentaquarks by Okiharu et al. [30], showing that the confining potential is provided by a fundamental string linking all the quarks and antiquarks. Cardoso et al. also confirmed this result with the Wilson loops for hybrids [31] and for three gluon glueballs [32]. Thus we assume that the confining component of the multiquark potential is

$$V_c(\mathbf{r}_i) = \sigma \sum_{i,a} r_{ia},$$
$$\mathbf{r}_{ia} = \mathbf{r}_i - \mathbf{r}_a, \tag{1}$$

where σ is the string tension, r_i is the position of the quark or antiquark Q_i, r_a is the position of the Fermat point F_a, and we use respectively Arab digits $i = 1, 2, 3, \ldots$ for the quarks (antiquarks) and Roman digits $a = \mathrm{I}, \mathrm{II}, \mathrm{III}, \ldots$ for the Fermat points. Thus the Fermat problem of finding the paths minimizing the total distance is equivalent to the physics problem of computing the multiquark potential. Notice that there are already some proposed experimental signals of tetraquarks, and the next generation of Hadronic Detectors may eventually observe multiquark hadrons.

The geometries of the strings of the first five multiquarks are depicted in Fig. 1. Eq. (1) and Fig. 1 extend the definition of the Fermat point of a triangle to the Fermat point of polygons in three dimensions with more points. With the present definition, where confinement is produced by fundamental strings, the strings meet in internal three-string vertices. The number of quarks can always be increased replacing a quark (antiquark) by a Fermat point and a diquark (di-antiquark). Thus the number of quarks (and antiquarks)

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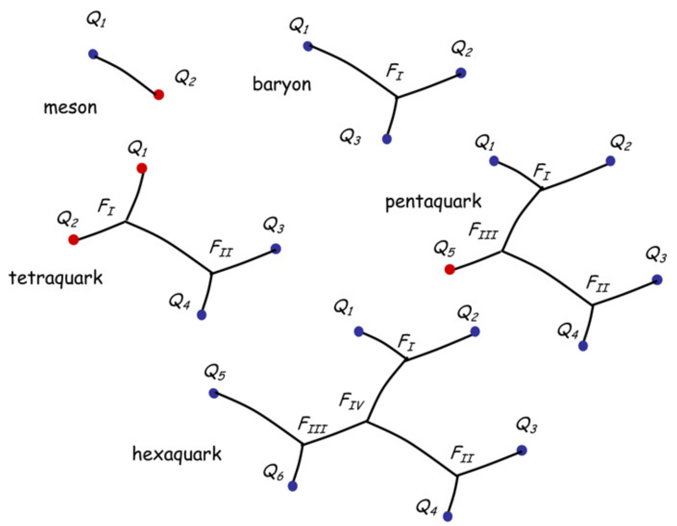

Fig. 1. The geometries of the string sections linking the first five multiquarks. Notice that the number of Fermat points F_a is $N-2$ where N is the number of quarks and antiquarks Q_i in the multiquark.

minus the number of Fermat points is a constant. Since in the meson and baryon this constant is 2, the number of Fermat points is $N-2$ where N is the number of quarks and antiquarks. Moreover in Eq. (1) we are only summing over distances between points linked by strings.

In general, the Fermat points depend on the choice of the string topology. In the case of the baryon there is only one way to attach the strings and thus the location of the only Fermat point do not depend on the location of the strings. In the case of the tetraquark, there is still only one possible way to attach the strings, but two different mesonic strings are also possible, corresponding to two different ways of forming mesons. For a larger number of quarks, say for the pentaquark, the hexaquark, or for a larger number of quarks, there are different ways of linking the quarks to fundamental strings. The decision on how to attach the strings, say with the flip-flop prescription of choosing the string configuration leading to a lower potential energy, must be taken externally to the method presented here.

For a meson (quark–antiquark system) the distance is trivially the quark–antiquark distance. For a baryon (three quark system) the problem was first solved geometrically by Fermat and by Torricelli. In the case of 3 quarks, the minimization of the potential in Eq. (1) implies that

$$\hat{r}_{1\,\mathrm{I}}+\hat{r}_{2\,\mathrm{I}}+\hat{r}_{3\,\mathrm{I}}=0, \tag{2}$$

and it is clear that the solution is that, either the triangle is not sufficiently acute, or the angles are all equal to 120°,

$$\widehat{\mathbf{r}_{1\,\mathrm{I}},\mathbf{r}_{2\,\mathrm{I}}}=\widehat{\mathbf{r}_{2\,\mathrm{I}},\mathbf{r}_{3\,\mathrm{I}}}=\widehat{\mathbf{r}_{3\,\mathrm{I}},\mathbf{r}_{1\,\mathrm{I}}}=120^\circ. \tag{3}$$

Due to the beauty of the triangles, and also to their simplicity, there are numerous geometry textbooks and articles on the Fermat–Torricelli point [1–6]. However, when the number of quarks increase, to tetraquarks, pentaquarks, etc., the geometric construction of the Fermat points becomes more and more difficult. Thus a numerical solution of this problem is welcome.

Here we address the computation of the total string distance and of the Fermat points where the different strings meet. In Section 2 we review briefly the geometrical methods leading to the Fermat points and to the total distances. In Section 3 we provide an iterative method, converging fast to the correct Fermat points and the total distances, relevant for the multiquark potentials. We detail the cases of the baryon, the tetraquark, the pentaquark and the hexaquark. In Section 4 we conclude.

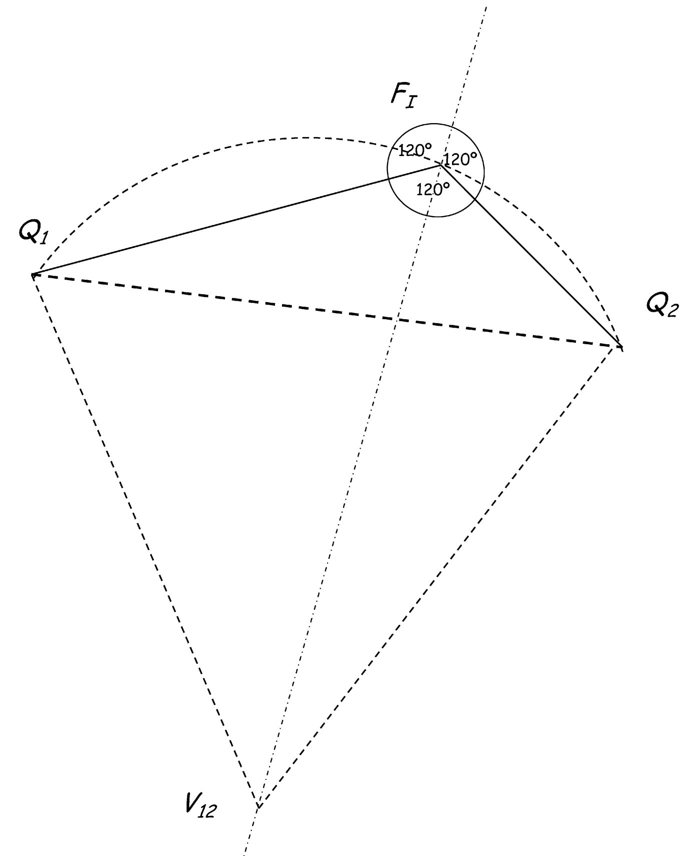

Fig. 2. A step in the geometric construction of the first Fermat point of an acute triangle. Starting from the segment Q_1Q_2, an equilateral triangle with vertex V_{12} is constructed. The Fermat point F_I belongs to the arc of circle centered in V_{12} and passing by Q_1 and Q_2.

2. Brief review of the geometrical method

In an acute triangle, the Fermat point F_I is the isogonic point, defined in Eq. (3). To construct the isogonic point, we start by the first pair of vertices Q_1 and Q_2, noticing that the set of points F_I with fixed angle $\widehat{Q_1F_\mathrm{I}Q_2}=120^\circ$ belong to an arc of circle. Moreover this circle is centred in the other vertex V_{12} of an equilateral triangle including Q_1 and Q_2. In Fig. 2 we show the 120° arc of circle, the equilateral triangle, and a segment including the points V_{12} and F_I. Notice that the other end of this segment forms with the segments Q_1F_I and $F_\mathrm{I}Q_2$ angles of 120°. Thus the isogonic point belongs this arc of circle. This point is at the intersection of the segments $V_{12}Q_3$, $V_{23}Q_1$ and $V_{31}Q_2$. The construction of first Fermat point F_I is illustrated in Fig. 3. It is very simple in a geometrical perspective, and it can be done just with a compass and a rule.

We now proceed with the tetraquark. This geometrical method can be extended to construct the two Fermat points F_I and F_II of a tetraquark. Notice that in the tetraquark we have four points, and thus in general the points Q_1, Q_2, Q_3 and Q_4 are not coplanar. Thus the vertices V_{12} and V_{34} are not, from the onset determined, only the circles where they belong are determined with the technique already used for the baryon. To determine the vertices, notice that the vertex V_{12} must be as far as possible from the segment Q_3Q_4 and that the vertex V_{34} must be as far as possible from the segment Q_1Q_2. Thus we find that the segment $V_{12}V_{34}$ must intersect the segment Q_1Q_2 and the segment Q_3Q_4. Then, once the segment $V_{12}V_{34}$ is determined, the Fermat points F_I and F_II are determined because the distances $V_{12}F_\mathrm{I}=Q_1Q_2$ and $V_{34}F_\mathrm{II}=Q_3Q_4$. This is illustrated in Fig. 4.