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Hamiltonian analysis of non-projectable modified $F(R)$ Hořava–Lifshitz gravity

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ABSTRACT

We study a version of the recently proposed modified $F(R)$ Hořava–Lifshitz gravity that abandons the projectability condition of the lapse variable. We discovered that the projectable version of this theory has a consistent Hamiltonian structure, and that the theory has interesting cosmological solutions which can describe the eras of accelerated expansion of the universe in a unified manner. The usual Hořava–Lifshitz gravity is a special case of our theory. Hamiltonian analysis of the non-projectable theory, however, shows that this theory has serious problems. These problems are compared with those found in the original Hořava–Lifshitz gravity. A general observation on the structure of the Poisson bracket of Hamiltonian constraints in all theories of the Hořava–Lifshitz type is made: in the resulting tertiary constraint the highest order spatial derivative of the lapse N is always of uneven order. Since the vanishing of the lapse ($N = 0$) is required by the preservation of the Hamiltonian constraints under time evolution, we conclude that the non-projectable version of the theory is physically inconsistent.

1. Introduction

Last year the so-called Hořava–Lifshitz theory of gravity was proposed [1] (see also [2,3]). This theory is a candidate for a quantum field theory of gravity that aims to provide an ultraviolet (UV) completion of General Relativity (GR). At short distances it describes interacting nonrelativistic gravitons. Hořava–Lifshitz gravity exhibits anisotropic scaling of space and time coordinates

$$\boldsymbol{x} \to b\boldsymbol{x}, \qquad t \to b^z t \tag{1.1}$$

with a dynamic critical exponent $z = 1, 2, 3, \ldots$. In the UV regime the value of the critical exponent z is chosen so that the gravitational coupling constant κ^2 is dimensionless. In $(D+1)$-dimensional space–time we have the scaling dimension $[\kappa^2] = z - D$. Thus the choice $z = D$ is argued to ensure that the theory is power-counting renormalizable. For the usual case of 3-dimensional space, $D = 3$, we choose $z = 3$.

The space–time manifold $\mathcal{M}$ is assumed to possess a foliation structure that enables one to define $\mathcal{M}$ as a union of space-like hypersurfaces Σ_t of constant time t. Due to the foliation the space–time is invariant under the foliation-preserving diffeomorphisms, whose infinitesimal generators are of the form

$$\delta\boldsymbol{x} = \zeta(t, \boldsymbol{x}), \qquad \delta t = f(t), \tag{1.2}$$

instead of the full diffeomorphism invariance of GR. For simplicity the topological structure of space–time is assumed to be such that every leaf Σ_t of the foliation is equivalent to a fixed manifold Σ: $\mathcal{M} \cong \mathbb{R} \times \Sigma$. The preferred foliation of $\mathcal{M}$ enables the inclusion of spatial covariant derivatives into the action, which improve the UV behaviour, while avoiding time derivatives higher than the second order, which are known to produce problematic ghosts.

At low energies and large distances the critical exponent is expected to flow to $z = 1$, so that the theory can coincide with GR. The Lorentz symmetry emerges at low energies as an accidental or approximate symmetry, but it is absent in the fundamental description.

The theory comes in two flavors, with or without the projectability condition that requires the lapse to depend only on the time coordinate, $N = N(t)$. The projectability condition is one of the features that makes the theory differ from GR. Note, however, that many

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solutions of GR, as well as of non-projectable Hořava–Lifshitz gravity, respect the condition $N = N(t)$ even thought the general theory does not.

In the original theory an additional symmetry, the condition of detailed balance, is assumed. It defines the potential part of the gravitational action in terms of a variation of a D-dimensional action on the spatial hypersurface with respect to the spatial metric. The purpose of the detailed balance condition is to reduce the number of independent couplings to 3 — otherwise there are 9 independent couplings. We, however, do not assume this condition, so all terms that have appropriate scaling properties and that are covariant under foliation-preserving diffeomorphisms can be included.

This theory has received a lot of attention and many potential problems have been discovered. Some of the problems are very serious. First it was found that GR is not recovered at large distances if the detailed balance condition is assumed [4,5]. A "phenomenologically viable" version of the theory without the detailed balance condition was soon introduced [6]. Due to the reduced diffeomorphism symmetry group there is an additional "half" scalar degree of freedom in Hořava–Lifshitz gravity. It has been shown to be strongly coupled at all scales by considering perturbations about a reasonable vacuum [7], regardless whether the detailed balance is assumed or not. This suggests that perturbative GR cannot be reproduced in Hořava–Lifshitz gravity [7], and that the theory could be ruled out by existing observations on the gravitational radiation of binary pulsars, which agree with linearized GR. The low-energy regime of the theory was further analyzed in Refs. [8,9] where problems with instability and strong coupling of the extra degree of freedom were found. Since then these problems have been confirmed in various papers. Although the non-projectable version of the theory may not give GR as the limit at large distances, some other scenarios, such as the chameleon, may solve this problem. A "healthy extension" of Hořava–Lifshitz gravity was proposed in Ref. [10] that is argued to be free from at least some of the pathologies of the original theory, since the extra scalar mode has a healthy quadratic action. This is achieved by adding terms that involve the spatial 3-vector $N^{-1}\nabla_i N$ into the action. We assume that the Hamiltonian takes the canonical form [see (2.19)] which rules out such terms. Hamiltonian formalism of the healthy extension has been studied in Ref. [11].

Renormalizability of the Hořava–Lifshitz gravity has been investigated beyond the power-counting scheme in Ref. [12].

When the projectability condition is assumed, the theory has a quite simple and consistent Hamiltonian structure. The algebra of constraints was shown to be closed for $z = 1, 2$ in Ref. [2], and this holds for a higher scaling exponent z as well. The Hamiltonian structure of Hořava–Lifshitz gravity without the projectability condition has been analyzed particularly in Refs. [13,14]. The non-projectable theory is physically inconsistent for generic couplings [14], including the case with detailed balance [13]. In the case of low-energy effective action a consistent set of constraints can be obtained by imposing an additional constraint ($\pi = 0$) [14–16].

Recently we proposed the modified $F(R)$ Hořava–Lifshitz gravity [17,18] that combines the interesting cosmological aspects of $F(R)$ gravity and the possible UV finiteness of Hořava–Lifshitz gravity. In particular, we demonstrated that the solution of spatially-flat FRW equation has two branches: one that coincides with the usual $F(R)$ gravity for a certain choice of parameters, and one that is totally new and typical only for Hořava–Lifshitz gravity. It was shown that unlike to standard Hořava–Lifshitz gravity, our $F(R)$ Hořava–Lifshitz gravity enables the possibility to unify the early-time inflation with the late-time acceleration in accord with the scenario of Ref. [19]. In this Letter we present the Hamiltonian analysis of the non-projectable version of this theory, where the lapse N depends also on the spatial coordinates: $N = N(t, \boldsymbol{x})$. Expectedly the Hamiltonian structure of this theory turns out to be more complex than in the projectable case. Our analysis should be of interest to everyone interested in the Hamiltonian formalism of gravity, and of modified gravity in particular.

2. Hamiltonian analysis

2.1. Action

We assume the ADM decomposition of space–time [20] (for reviews and mathematical background, see [21]). The metric tensor of space–time is decomposed in terms of the ADM variables as

$$ {}^4g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -\left(N^2 - N_i N^i\right)\mathrm{d}t^2 + N_i\left(\mathrm{d}t\,\mathrm{d}x^i + \mathrm{d}x^i\,\mathrm{d}t\right) + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{2.1} $$

where N is the lapse, N^i is the shift vector, g_{ij} is the spatial metric tensor, and $x^i, i = 1, 2, 3$ are spatial coordinates on the $t =$ constant hypersurface Σ_t. The covariant derivatives defined by the metric tensors ${}^4g_{\mu\nu}$ and g_{ij} are denoted by $\nabla^{(4)}_\mu$ and ∇_i, respectively. The extrinsic curvature of the hypersurface Σ_t is

$$ K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - 2\nabla_{(i}N_{j)}\right), \tag{2.2} $$

where the dot denotes the derivative with respect to time t. The scalar associated to the extrinsic curvature is denoted by $K = g^{ij}K_{ij}$. The (intrinsic) curvature of the space Σ_t is defined by the spatial metric g_{ij} in the usual manner. The natural invariant volume element of space–time is decomposed

$$ \mathrm{d}^4x\sqrt{-{}^4g} = \mathrm{d}t\,\mathrm{d}^3\boldsymbol{x}\sqrt{g}N. \tag{2.3} $$

The action of the non-projectable version of the modified $F(R)$ Hořava–Lifshitz gravity is defined similarly as in the projectable case [17]:

$$ S_F = \frac{1}{\kappa^2}\int \mathrm{d}t\,\mathrm{d}^3\boldsymbol{x}\sqrt{g}NF\left({}^4\tilde{R}\right), $$
$$ {}^4\tilde{R} \equiv K_{ij}K^{ij} - \lambda K^2 + 2\mu\nabla^{(4)}_\mu\left(n^\mu\nabla^{(4)}_\nu n^\nu - n^\nu\nabla^{(4)}_\nu n^\mu\right) - \mathcal{L}_R(g_{ij}). \tag{2.4} $$

Here λ and μ are constants, n^μ is the unit normal to the spatial hypersurfaces Σ_t, and $\mathcal{L}_R(g_{ij})$ is a function of the three-dimensional metric g_{ij} and the covariant derivatives ∇_i defined by this metric. The crucial difference compared to the theory we proposed and