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Quasinormal spectrum and the black hole membrane paradigm

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ABSTRACT

The membrane paradigm approach to black hole physics introduces the notion of a stretched horizon as a fictitious time-like surface endowed with physical characteristics such as entropy, viscosity and electrical conductivity. We show that certain properties of the stretched horizons are encoded in the quasinormal spectrum of black holes. We compute analytically the lowest quasinormal frequency of a vector-type perturbation for a generic black hole with a translationally invariant horizon (black brane) in terms of the background metric components. The resulting dispersion relation is identical to the one obtained in the membrane paradigm treatment of the diffusion on stretched horizons. Combined with the Buchel–Liu universality theorem for the membrane's diffusion coefficient, our result means that in the long wavelength limit the black brane spectrum of gravitational perturbations exhibits a universal, purely imaginary quasinormal frequency. In the context of gauge–gravity duality, this provides yet another (third) proof of the universality of shear viscosity to entropy density ratio in theories with gravity duals.

1. Introduction

In the membrane paradigm approach to black holes, the horizon is replaced by a time-like surface (the stretched horizon) located infinitesimally close to the true mathematical horizon [1,2]. The stretched horizon behaves as an effective membrane endowed with physical properties such as electrical conductivity and viscosity. Furthermore, in the case of black branes (black holes with translationally invariant horizons) it was shown [3] that the charge density j^0 defined on the stretched horizon by the standard membrane paradigm construction [4] obeys the diffusion equation

$$\partial_t j^0 = D\nabla^2 j^0 \tag{1}$$

in the long-wavelength (hydrodynamic) limit. (The translational invariance of the horizon guarantees the existence of the hydrodynamic limit.) The corresponding dispersion relation has the form

$$\omega = -iDq^2, \tag{2}$$

where q is the momentum along the stretched horizon. For a generic black brane metric, the diffusion coefficient D can be determined explicitly in terms of the metric components [3]. Moreover, for the so-called shear mode of the *metric* fluctuation, one can write an effective Maxwell action and an effective charge density on the stretched horizon satisfying Eq. (1) with the shear mode damping constant determined by the ambient metric. A number of examples considered in [3] suggested that (1) the membrane's shear mode damping constant was equal to $1/4\pi T$ independently of the background metric and (2) the membrane's diffusion and the shear mode damping constants coincided, respectively, with the appropriate diffusion and damping constants of the currents and stress-energy tensors in the dual theory computed via the AdS/CFT correspondence. Using Einstein's equations, Buchel and Liu [5] proved that the suggestion (1) is universally true for the class of metrics considered in [3]. (An alternative proof of the universality of the membrane's coefficient employing the Lorentz boost of the black brane metric can be found in Section 6 of the review [6].) This proof, however, cannot be viewed as the proof of the universality of the shear viscosity to entropy density ratio in the dual field theory without the proof of the suggestion (2). One of the goals of the present Letter is to supply this missing link by proving the suggestion (2). To prove it, we need either to understand why the membrane paradigm appears to "know" about the gauge/gravity duality or to derive the generic formulas of [3] for the diffusion and damping constants by holographic means only, without appealing to membrane paradigm constructions. In this Letter, we focus on the latter alternative.

More generally, our goal is to show that at least some of the properties of the stretched horizon are encoded in the quasinormal spectrum of the corresponding black hole (brane). (For a review on quasinormal modes see e.g. [7].) We compute analytically the lowest quasinormal frequency of a vector-type fluctuation in the background of a black brane and show that it is of the form (2)

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with the coefficient D identical to the diffusion coefficient computed in [3] using the "membrane paradigm" approach. Since in the gauge/gravity duality the quasinormal spectrum of bulk field fluctuations is identified with the poles of the retarded correlators of operators dual to the fluctuations [8], our result proves the suggestion (2) and, together with Ref. [5] (or Ref. [6]), provides yet another proof of the universality of the shear viscosity to entropy density ratio in thermal field theories in the regime described by dual classical gravity. (The other two proofs are Refs. [9,10]. Clearly, all three approaches are interrelated.)

2. Quasinormal spectrum of a $U(1)$ fluctuation in a black brane background

A black p-brane is represented by the metric

$$ds^2 = G_{tt}(r)\,dt^2 + G_{rr}(r)\,dr^2 + G_{xx}(r)\sum_{i=1}^{p}\left(dx^i\right)^2. \tag{3}$$

Such metrics typically result from a dimensional reduction of higher-dimensional supergravity solutions. As a guide, one can have in mind the near-extremal black three-brane solution of type II supergravity dimensionally reduced on a five-sphere

$$ds^2 = \frac{r^2}{R^2}\left(-f\,dt^2 + dx^2 + dy^2 + dz^2\right) + \frac{R^2}{r^2 f}dr^2,$$
$$f = 1 - \frac{r_0^4}{r^4}, \tag{4}$$

but our discussion will be quite general. We assume that the metric (3) has a translationally-invariant event horizon at $r = r_0$ that extends in p spatial dimensions parametrized by the coordinates x^i. It will be convenient to introduce a dimensionless coordinate $u = r_0^2/r^2$ that maps the semi-infinite interval $r \in [r_0, \infty)$ into a finite one, $u \in [0, 1]$. The metric becomes

$$ds^2 = g_{tt}(u)\,dt^2 + g_{uu}(u)\,du^2 + g_{xx}(u)\sum_{i=1}^{p}\left(dx^i\right)^2, \tag{5}$$

where the components are related to the ones in Eq. (3) by trivial redefinitions [3]. We assume that near the horizon, i.e. in the limit $u \to 1$, the components g_{tt}, g_{uu}, g_{xx} behave as

$$g_{tt} = -(1-u)\gamma_0 + O(1-u), \tag{6}$$
$$g_{uu} = \frac{\gamma_u}{1-u} + O(1), \tag{7}$$
$$g_{xx} = O(1), \tag{8}$$

where γ_0 and γ_u are positive constants. We also introduce a thermal factor function

$$f(u) = -g_{tt}(u)/g_{xx}(u). \tag{9}$$

The function $f(u)$ has a simple zero at $u = 1$. The Hawking temperature associated with the background (5) is

$$T = \frac{1}{4\pi}\sqrt{\frac{\gamma_0}{\gamma_u}}. \tag{10}$$

In our example of the black three-brane solution (4), the metric in the new coordinates is given by

$$ds^2 = \frac{(\pi T R)^2}{u}\left(-f(u)\,dt^2 + dx^2 + dy^2 + dz^2\right) + \frac{R^2}{4u^2 f(u)}du^2,$$
$$f = 1 - u^2. \tag{11}$$

Consider now fluctuations of a $U(1)$ field $A_\mu(u,t,x)$ in the background (5). This field can be viewed e.g. as a graviphoton of the dimensional reduction. Translational invariance of the horizon implies that the fluctuation can be taken to be proportional to $e^{-i\omega t + i\mathbf{qx}}$, and we choose the spatial momentum to be directed along $x \equiv x^p$.

In the gauge $A_u = 0$, Maxwell's equations $\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0$ for the components $A_t(u,t,x)$ and $A_x(u,t,x)$ read

$$g^{tt}\omega A_t' - q g^{xx} A_x' = 0, \tag{12}$$
$$\partial_u\left(\sqrt{-g}\,g^{tt}g^{uu}A_t'\right) - \sqrt{-g}\,g^{tt}g^{xx}\left(\omega q A_x + q^2 A_t\right) = 0, \tag{13}$$
$$\partial_u\left(\sqrt{-g}\,g^{xx}g^{uu}A_t'\right) - \sqrt{-g}\,g^{tt}g^{xx}\left(\omega q A_t + \omega^2 A_x\right) = 0, \tag{14}$$

where prime denotes the derivative with respect to u. All other components of $A_\mu(u,t,x)$ decouple, and thus can be consistently set to zero.

For a gauge-invariant combination $E_x = \omega A_x + q A_t$ (the component of the electric field parallel to the brane) the system (12)–(14) yields the following equation

$$E_x'' + \left[\frac{\mathfrak{w}^2 f'}{f(\mathfrak{w}^2 - \mathfrak{q}^2 f)} + \partial_u \log\left(\sqrt{-g}\,g^{tt}g^{uu}\right)\right]E_x'$$
$$+ \frac{(2\pi T)^2 g^{xx}}{f g^{uu}}\left(\mathfrak{w}^2 - \mathfrak{q}^2 f\right)E_x = 0, \tag{15}$$

where $\mathfrak{w} = \omega/2\pi T$, $\mathfrak{q} = q/2\pi T$. The differential equation (15) has a singular point at $u = 1$ with the exponents $\alpha_\pm = \pm i\mathfrak{w}/2$ corresponding to the waves emerging from and disappearing into the horizon. Imposing the incoming wave boundary condition at the horizon, one can write the solution as

$$E_x(u) = f^{-i\mathfrak{w}/2}F(u), \tag{16}$$

where $F(u)$ is regular at $u = 1$. At spatial infinity, $u = 0$, we impose the Dirichlet boundary condition $E_x(0) = 0$.

We are interested in computing the quasinormal spectrum of the fluctuation E_x subject to the boundary conditions stated above. Generically, we expect the spectrum to consist of an infinite tower $\omega_n = \omega_n(q)$ of the discrete complex frequencies. The lowest frequency, $\omega_0(q)$, can have a finite gap as $q \to 0$, or be gapless, $\lim_{q\to 0}\omega_0(q) = 0$. We will now show that the frequency of the vector-like fluctuation $E_x(u,t,x)$ is in fact gapless, and compute its value in the limit of small ω, q.

An analytic solution to Eq. (15) in the limit $\mathfrak{w} \ll 1$, $\mathfrak{q} \ll 1$ can be easily found. Introducing a book-keeping parameter λ and rescaling $\mathfrak{w} \to \lambda\mathfrak{w}$, $\mathfrak{q} \to \lambda\mathfrak{q}$, one can obtain a perturbative solution in the form

$$E_x(u, \mathfrak{w}, \mathfrak{q}) = f^{-i\mathfrak{w}/2}\left(F_0(u) + \lambda F_1(u) + O\left(\lambda^2\right)\right), \tag{17}$$

where each of the functions $F_i(u)$ obeys an equation derived from Eq. (15). The equation for $F_0(u)$ has a generic solution

$$F_0 = C_0 + C_1\int \frac{(\mathfrak{w}^2 - \mathfrak{q}^2 f)\,du}{f\sqrt{-g}\,g^{tt}g^{uu}}, \tag{18}$$

where C_0, C_1 are integration constants. The integral in Eq. (18) is logarithmically divergent at $u = 1$. Since by construction F_0 is a regular function, we must put $C_1 = 0$. The function $F_1(u)$ obeys an inhomogeneous equation whose regular at $u = 1$ solution is given by

$$F_1(u) = -\frac{i\mathfrak{w}C_0}{2}\log f - \frac{iC_0\sqrt{-g(1)}\,f'(1)}{2\mathfrak{w}\gamma_0\gamma_u}\int \frac{(\mathfrak{w}^2 - \mathfrak{q}^2 f)\,du}{f\sqrt{-g}\,g^{tt}g^{uu}}. \tag{19}$$

Given these explicit solutions, we use the Dirichlet condition $E_x(0) = 0$ to obtain the following equation for $\mathfrak{w}$

$$\mathfrak{w} - i\mathfrak{q}^2\frac{\sqrt{-g(1)}\,f'(1)}{2\gamma_0\gamma_u}\int_0^1 \frac{du}{\sqrt{-g}\,g^{tt}g^{uu}} + O\left(\mathfrak{w}^2\right) = 0. \tag{20}$$