Physics Letters B 663 (2008) 132–135

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Odd scalar curvature in anti-Poisson geometry

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ARTICLE INFO

Article history:
Received 2 January 2008
Accepted 31 March 2008
Available online 7 April 2008
Editor: M. Cvetič

PACS:
02.40.-k
03.65.Ca
04.60.Gw
11.10.-z
11.10.Ef
11.15.Bt

Keywords:
BV field–antifield formalism
Odd Laplacian
Anti-Poisson geometry
Semidensity
Connection
Odd scalar curvature

ABSTRACT

Recent works have revealed that the recipe for field–antifield quantization of Lagrangian gauge theories can be considerably relaxed when it comes to choosing a path integral measure ρ if a zero-order term ν_ρ is added to the Δ operator. The effects of this odd scalar term ν_ρ become relevant at two-loop order. We prove that ν_ρ is essentially the odd scalar curvature of an arbitrary torsion-free connection that is compatible with both the anti-Poisson structure E and the density ρ. This extends a previous result for non-degenerate antisymplectic manifolds to degenerate anti-Poisson manifolds that admit a compatible two-form.

1. Introduction

The main purpose of this Letter is to report on new geometric insights into the field–antifield formalism. In general, the field–antifield formalism [1–3] is a recipe for constructing Feynman rules for Lagrangian field theories with gauge symmetries. The field–antifield formalism is in principle able to handle the most general gauge algebra, i.e. open gauge algebras of reducible type. The input is usually a local relativistic field theory, formulated via a classical action principle in a geometric configuration space. In the field–antifield scheme, the original field variables are extended with various stages of ghosts, antighosts and Lagrange multipliers—all of which are then further extended with corresponding antifields; the gauge symmetries are encoded in a nilpotent Fermionic BRST symmetry [4,5]; and the original action is deformed into a BRST-invariant master action, whose Hessian has the maximal allowed rank. The full quantum master action

$$W = S + \sum_{n=1}^{\infty} \hbar^n M_n \tag{1.1}$$

is determined recursively order by order in $\hbar$ from a consistent set of quantum master equations

$$(S, S) = 0, \tag{1.2}$$

$$(M_1, S) = i(\Delta_\rho S), \tag{1.3}$$

$$(M_2, S) = i(\Delta_\rho M_1) + \nu_\rho - \frac{1}{2}(M_1, M_1), \tag{1.4}$$

$$(M_n, S) = i(\Delta_\rho M_{n-1}) - \frac{1}{2}\sum_{r=1}^{n-1}(M_r, M_{n-r}), \quad n \geqslant 3. \tag{1.5}$$

Here $(\cdot,\cdot)$ is the antibracket (or anti-Poisson structure), Δ_ρ is the odd Laplacian and ν_ρ is an odd scalar, which become relevant in perturbation theory at loop order 0, 1, and 2, respectively. It has only recently been realized that the field–antifield formalism can consistently accommodate a non-zero ν_ρ term, thereby providing a more flexible framework for field–antifield quantization [6–8].

The classical master equation (1.2) is a generalization of Zinn-Justin's equation [9], which allows to set up consistent renormalization (if the field theory is renormalizable). If the theory is not anomalous at the one-loop level, there will exist a local solution M_1 to the next Eq. (1.3), and so forth. Although the field–antifield formalism in its basic form is only a formal scheme—i.e. particularly, it assumes that results from finite-dimensional analysis are

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0370-2693/$ – see front matter
doi:10.1016/j.physletb.2008.03.066

directly applicable to field theory, which has infinitely many degrees of freedom—it has nevertheless been successfully applied to a large variety of physical models. It has mainly been used in a truncated form of the full set of quantum master Eqs. (1.2)–(1.5), where all the following quantities

$$(S,S), (\Delta_\rho S), \nu_\rho, M_1, M_2, M_3, \ldots, \tag{1.6}$$

are set identically equal to zero. One can for instance mention the AKSZ paradigm [10,11] as a broad example that uses the truncated field–antifield formalism (1.6) to quantize supersymmetric topological field theories [12–15]. Currently, very few scientific works describe solutions with non-zero M_n's, primarily due to the singular nature of the odd Laplacian Δ_ρ in field theory (again because of the infinitely many degrees of freedom). Nevertheless, it should be fruitful to study generic solutions of the full quantum master equation. See the original paper [1] for an interesting solution with $M_1 \neq 0$. Finally, it has in many cases been explicitly checked that the field–antifield formalism produces the same result as the Hamiltonian formulation [16–18]. The formalism has also influenced work in closed string field theory [19] and several branches of mathematics. The geometry behind the field–antifield formalism was further clarified in Refs. [20–23].

In this Letter we shall only explicitly consider the case of finitely many variables. Our main result concerns the odd scalar ν_ρ, which is a certain function of the anti-Poisson structure E^{AB} and the density ρ, cf. Eq. (6.1) below. It turns out that ν_ρ has a geometric interpretation as (minus $1/8$ times) the odd scalar curvature R of any connection ∇ that satisfies three conditions; namely that ∇ is (1) anti-Poisson, (2) torsion-free and (3) ρ-compatible. This is a rather robust conclusion as we shall prove in this Letter that it even holds for degenerate antibrackets. (Degenerate anti-Poisson structures appear naturally from for instance the Dirac antibracket construction for antisymplectic second-class constraints [7,21,24,25].)

2. Anti-Poisson structure E^{AB}

An *anti-Poisson* structure is by definition a possibly degenerate $(2,0)$ tensor field E^{AB} with upper indices that is Grassmann-odd

$$\varepsilon\big(E^{AB}\big) = \varepsilon_A + \varepsilon_B + 1, \tag{2.1}$$

that is skewsymmetric

$$E^{AB} = -(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} E^{BA}, \tag{2.2}$$

and that satisfies the Jacobi identity

$$\sum_{\text{cycl.}A,B,C} (-1)^{(\varepsilon_A+1)(\varepsilon_C+1)} E^{AD}\big(\overrightarrow{\partial^l_D} E^{BC}\big) = 0. \tag{2.3}$$

3. Compatible two-form E_{AB}

In general, an anti-Poisson manifold could have singular points where the rank of E^{AB} jumps, and it is necessary to impose a regularity criterion to proceed. We shall here assume that the anti-Poisson structure E^{AB} admits a compatible two-form field E_{AB}, i.e. that there exists a two-form field E_{AB} with lower indices that is Grassmann-odd

$$\varepsilon(E_{AB}) = \varepsilon_A + \varepsilon_B + 1, \tag{3.1}$$

that is skewsymmetric

$$E_{AB} = -(-1)^{\varepsilon_A\varepsilon_B} E_{BA}, \tag{3.2}$$

and that is *compatible* with the anti-Poisson structure in the sense that

$$E^{AB}E_{BC}E^{CD} = E^{AD}, \tag{3.3}$$

$$E_{AB}E^{BC}E_{CD} = E_{AD}. \tag{3.4}$$

This is a relatively mild requirement, which is always automatically satisfied for a Dirac antibracket on antisymplectic manifolds with antisymplectic second-class constraints [7,21,24,25]. Note that the two-form E_{AB} is neither unique nor necessarily closed. One can define a $(1,1)$ tensor field as

$$P^A{}_C \equiv E^{AB}E_{BC}, \tag{3.5}$$

or equivalently,

$$P_A{}^C \equiv E_{AB}E^{BC} = (-1)^{\varepsilon_A(\varepsilon_C+1)} P^C{}_A. \tag{3.6}$$

It then follows from either of the compatibility relations (3.3) and (3.4) that $P^A{}_B$ is an idempotent

$$P^A{}_B P^B{}_C = P^A{}_C. \tag{3.7}$$

4. The Δ_E operator

An anti-Poisson structure with a compatible two-form field E_{AB} gives rise to a Grassmann-odd, second-order Δ_E operator that takes semidensities to semidensities. It is defined in arbitrary coordinates as [7]

$$\Delta_E \equiv \Delta_1 + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{8} - \frac{\nu^{(3)}}{24} + \frac{\nu^{(4)}}{24} + \frac{\nu^{(5)}}{12}, \tag{4.1}$$

where Δ_1 is the odd Laplacian

$$\Delta_\rho \equiv \frac{(-1)^{\varepsilon_A}}{2\rho}\overrightarrow{\partial^l_A}\rho E^{AB}\overrightarrow{\partial^l_B}, \tag{4.2}$$

with $\rho = 1$, and where

$$\nu^{(1)} \equiv (-1)^{\varepsilon_A}\big(\overrightarrow{\partial^l_B}\overrightarrow{\partial^l_A}E^{AB}\big), \tag{4.3}$$

$$\nu^{(2)} \equiv (-1)^{\varepsilon_A\varepsilon_C}\big(\overrightarrow{\partial^l_D}E^{AB}\big)E_{BC}\big(\overrightarrow{\partial^l_A}E^{CD}\big), \tag{4.4}$$

$$\nu^{(3)} \equiv (-1)^{\varepsilon_B}\big(\overrightarrow{\partial^l_A}E_{BC}\big)E^{CD}\big(\overrightarrow{\partial^l_D}E^{BA}\big), \tag{4.5}$$

$$\nu^{(4)} \equiv (-1)^{\varepsilon_B}\big(\overrightarrow{\partial^l_A}E_{BC}\big)E^{CD}\big(\overrightarrow{\partial^l_D}E^{BF}\big)P_F{}^A, \tag{4.6}$$

$$\begin{aligned}\nu^{(5)} &\equiv (-1)^{\varepsilon_A\varepsilon_C}\big(\overrightarrow{\partial^l_D}E^{AB}\big)E_{BC}\big(\overrightarrow{\partial^l_A}E^{CF}\big)P_F{}^D\\ &= (-1)^{(\varepsilon_A+1)\varepsilon_B}E^{AD}\big(\overrightarrow{\partial^l_D}E^{BC}\big)\big(\overrightarrow{\partial^l_C}E_{AF}\big)P^F{}_B.\end{aligned} \tag{4.7}$$

It is shown in Ref. [7] that the Δ_E operator defined in Eq. (4.1) does not depend on the choice of local coordinates, it does not depend on the choice of compatible two-form field E_{AB}, and it does map semidensities into semidensities. Moreover, the Jacobi identity (2.3) precisely ensures that Δ_E is nilpotent

$$\Delta_E^2 = \frac{1}{2}[\Delta_E, \Delta_E] = 0. \tag{4.8}$$

Earlier works on the Δ_E operator include Refs. [6,25–29].

5. The Δ operator

Classically, the field–antifield formalism is governed by the anti-Poisson structure E^{AB}, or equivalently, the antibracket

$$(f,g) \equiv \big(f\overleftarrow{\partial^r_A}\big)E^{AB}\big(\overrightarrow{\partial^l_B}g\big) = -(-1)^{(\varepsilon_f+1)(\varepsilon_g+1)}(g,f). \tag{5.1}$$

Quantum mechanically, the field–antifield recipe instructs one to choose an arbitrary path integral measure ρ, and to use it to build a nilpotent, Grassmann-odd, second-order Δ operator that takes scalar functions into scalar functions. It is natural to build the Δ operator by conjugating the Δ_E operator (4.1) with appropriate square roots of the density ρ as follows:

$$\Delta \equiv \frac{1}{\sqrt{\rho}}\Delta_E\sqrt{\rho}. \tag{5.2}$$