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Lifetime of unstable hairy black holes

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Abstract

During the last two decades solutions of black holes with various types of "hair" have been discovered. Remarkably, it has been established that many of these hairy black holes are unstable—under small perturbations the hair may collapse. While the static sector of theories admitting hair is well explored by now, our picture of the dynamical process of hair-shedding is still incomplete. In this Letter we provide an important ingredient of the nonlinear dynamics of hair collapse: we derive a universal lower bound on the lifetime of hairy black holes. It is also shown that the amount of hair outside of a black-hole horizon should be fundamentally bounded.

Wheeler's dictum "a black hole has no hair" [1] has played a major role in the development of black-hole physics [2,3]. The conjecture implies that black holes are fundamental objects: they should be described by only a few parameters, very much like atoms in quantum mechanics. The fact that stationary black holes are specified by conserved charges which are associated with a Gauss-like law (mass, charge, and angular momentum) was proven explicitly in Einstein vacuum theory and Einstein–Maxwell theory [4]. Early no-hair theorems also excluded scalar [5], massive vector [6], and spinor [7] fields from the exterior of a stationary black hole.

However, later day developments in particle physics have lead to the somewhat surprising discovery of various types of "hairy" black holes, the first of which were the "colored black holes" [8]. These are static black hole solutions of the Einstein–Yang–Mills (EYM) theory that require for their complete specification not only the value of the mass, but also an additional integer (counting the number of nodes of the function characterizing the Yang–Mills field outside the horizon) that is not, however, associated with any conserved charge. Following the discovery of the colored black holes, other hairy solutions have been found [9–11].

Remarkably, many of these hairy black-hole solutions were proven to be unstable [12–14]. This implies that under small perturbations the black hole would lose its hair. Numerical studies of hairy black-hole spacetimes [15,16] have revealed two distinct mechanisms of losing unstable hair: the hair may be dispersed to infinity, leaving the original black hole virtually unchanged, or it may collapse into the black hole, in which case the original black hole gets bigger.

While the *static* sector of theories admitting hair is well understood by now, we still luck a complete picture of the *dynamical* processes by which perturbed, unstable hairy black holes shed their hair. In particular, the fact that many of the hairy black-hole solutions were found to be dynamically unstable promotes the fundamental question: what is the lifetime of an unstable hairy black hole?

In this Letter we derive a universal lower bound on the dynamical lifetime, τ_{hair}, of perturbed, unstable hairy black-hole solutions. Our derivation is based on standard results of information theory and universal thermodynamic considerations.

A fundamental question in quantum information theory is what is the maximum rate at which information/entropy [17]

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may be changed by a signal of duration τ and energy $\Delta\mathcal{E}$. The answer to this question is given by the Bekenstein–Bremermann relation [18–20]:

$$\frac{\Delta S}{\tau} \leqslant \pi \Delta\mathcal{E}/\hbar. \tag{1}$$

(We use gravitational units in which $G = c = k_B = 1$.) We shall now use this relation in the context of black-hole dynamics.

Accretion of perturbed, unstable hair into a black hole would increase the black-hole mass[1] by an amount $\Delta\mathcal{E}$ ($\Delta\mathcal{E} \leqslant \mathcal{M}_{\rm hair}$, where $\mathcal{M}_{\rm hair}$ is the mass of the black-hole hair outside the horizon). As a consequence its area (entropy) would also increase. This implies that information/entropy is actually flowing into the black hole during this dynamical process.

One may now derive an upper bound on the dynamical rate at which the hair collapse (or equivalently, an upper bound on the rate at which the black-hole entropy increases due to hair "swallowing"). For simplicity we focus here on spherically symmetric spacetimes. The mass $m(r_H)$ contained within the black-hole horizon is related to the horizon's radius by $m(r_H) = \frac{1}{2}r_H$ [3], and the mass of the outside hair is given by $\mathcal{M}_{\rm hair} = \mathcal{M}_{\rm total} - \frac{1}{2}r_H$. It is worth mentioning that [21] provides a detailed discussion about the relation between the bare mass of the black hole (without hair), the mass of the hair, and the ADM mass of the black-hole spacetime.

An increase $\Delta\mathcal{E}$ in the mass contained within the black hole (due to hair swallowing) would increase the black-hole radius from r_H to $r_H + 2\Delta\mathcal{E}$. Taking cognizance of the black-hole area-entropy relation $S_{\rm BH} = A/4\hbar = \pi r_H^2/\hbar$ [2], one finds that the corresponding change in black-hole entropy is given by $\Delta S_{\rm BH} = \pi[(r_H + 2\Delta\mathcal{E})^2 - r_H^2]/\hbar$. Substituting

$$\Delta S_{\rm BH} = 4\pi \Delta\mathcal{E}(r_H + \Delta\mathcal{E})/\hbar, \tag{2}$$

into Eq. (1), we obtain a universal lower bound on the lifetime $\tau_{\rm hair}$ (as measured by a static observer at infinity) of a perturbed, unstable hairy black hole:

$$\tau_{\rm hair} \geqslant 4(r_H + \Delta\mathcal{E}) \geqslant 4r_H. \tag{3}$$

It is worth nothing that Núñez et al. [3] have proved a very nice theorem according to which the "hairosphere", the region where the nonlinear behavior of the black-hole hair is present, must extend beyond $\frac{3}{2}r_H$. This would imply that the collapse time of perturbed, unstable hair into the black hole is roughly bounded by $\geqslant \frac{1}{2}r_H$. The universal bound, Eq. (3), is (at least) an order of magnitude stronger. This dynamical bound, $\tau_{\rm hair} \geqslant 4(r_H + \Delta\mathcal{E}) \geqslant 4r_H$, is in fact complimentary to the spatial bound $r_{\rm hair} \geqslant \frac{3}{2}r_H$ of [3].

We would like to emphasize here that hairy black-hole solutions are characterized by several different length/time scales:

- The radius of the black-hole horizon, r_H.
- The mass of the outside hair, $\mathcal{M}_{\rm hair}$.[2]
- The inverse of the black-hole temperature $T_{\rm BH}$ (proportional to the surface gravity).
- In some hairy solutions there is additional length scale set by the dimensional coupling constant of the theory (for example, the reciprocal of the YM coupling constant, g) [9]. Likewise, in theories with massive hairy fields [11] there is another characteristic scale—the Compton length of the field, $\hbar$/mass.

The diversity of these independent length/time scales which characterize hairy black-hole solutions, implies that any attempt to obtain a bound on the lifetime of hairy black holes from naive dimensionality considerations would be too presumptuous—there are simply too many independent scales and dimensional parameters in the problem. In principle, the characteristic dynamical timescale could have turned out to be any complicated combination of the above mentioned parameters. However, the analytically derived bound, $\tau_{\rm hair} \geqslant 4r_H$,[3] is remarkably simple and universal in the sense that it is independent of all other dimensional parameters present in the theory.

Testing the bound. The early stages of a full, nonlinear collapse of unstable hair can be described by linearized perturbations. Thus the total lifetime of the unstable hairy black hole is bounded from below by the characteristic time which describes the early growth of linear perturbations. (The total lifetime is actually longer and includes the phase of nonlinear dynamics which follows the linear regime.) The linear instability time is given by the reciprocal of the eigenvalue σ corresponding to the unstable mode of the hairy black hole (a linearized perturbation mode which grows according to $\sim e^{\sigma t}$). One therefore concludes that the total lifetime of an unstable hairy black is bounded by $\tau_{\rm hair} \geqslant 1/\sigma$. Thus, a sufficient (but not a necessary) condition for the validity of the universal bound, Eq. (3), is that the instability eigenvalue σ is bounded by

$$\sigma \leqslant \sigma_{\max} \equiv \left[4(r_H + \Delta\mathcal{E})\right]^{-1} \leqslant (4r_H)^{-1}. \tag{4}$$

As an example, we display in Table 1 the numerically computed (see Ref. [22]) eigenvalues σ corresponding to the unstable mode of the canonical EYM (colored) hairy black holes. We also present the ratio $\sigma/\sigma_{\max}$. We first consider the weak version of the bound, in which case we take $\Delta\mathcal{E} \to 0$. One finds $\sigma/\sigma_{\max}^{\rm weak} < 1$ for all black holes, in accord with our analytical prediction.

It has been shown [15,16] that, depending on the initial perturbation most of the unstable hair may collapse into the black hole. We should therefore check the validity of our bound in its strong version, in which case we take $\Delta\mathcal{E} = \mathcal{M}_{\rm hair}$. We display the ratio $\sigma/\sigma_{\max}^{\rm strong}$ in Table 1, from which one learns that the unstable hairy black holes conform to the bound Eq. (4) even in its

[1] We assume the validity of the weak energy condition, which insures the positivity of the hair's mass.

[2] We emphasize that the mass of the black-hole hair, $\mathcal{M}_{\rm hair}$ does not scale linearly with its radius, r_H, see e.g., [22]. These two quantities therefore determine different length/time scales.

[3] The bound can be made stronger, $\tau_{\rm hair} \geqslant 4(r_H + \mathcal{M}_{\rm hair})$, as discussed below.