Physics Letters B 751 (2015) 227–232

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Entropy discrepancy and total derivatives in trace anomaly

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ARTICLE INFO

Article history:
Received 12 October 2015
Accepted 13 October 2015
Available online 21 October 2015
Editor: M. Cvetič

ABSTRACT

In this note we address the discrepancy found by Hung, Myers and Smolkin between the holographic calculation of entanglement entropy (using the Jacobson–Myers functional for the holographic minimal surface) and the CFT trace anomaly calculation if one uses the Wald prescription to compute the entropy in six dimensions. As anticipated in our previous work [1] the discrepancy originates entirely from a total derivative term present in the trace anomaly in six dimensions.

1. Introduction

The systematic study of entropy associated to the gravitational action has been initiated by works of Wald [2] and was initially motivated by applications to the entropy of black hole horizons [3]. It was realized that the formula for computing the entropy is in a certain correspondence with the terms in the gravitational action. The usual area law for black hole horizons in the Einstein theory of gravity is necessarily modified as soon as the action includes terms of higher power in curvature. The conical singularity method introduced in [4] has sharpened this correspondence and provided an efficient algorithm to compute the entropy in a way which does not a priori require the metric to satisfy any field equations. For the Killing horizons this off-shell method is in a complete agreement with the Wald prescription although the latter requires the metric to be on-shell, i.e. satisfy some gravitational equations. This method is purely geometrical. It explores the distributional nature of the conical singularities. Later it was realized that the conical singularity method has a much wider applicability and can be used very efficiently to compute the entanglement entropy associated to an arbitrary surface Σ, not necessarily a black hole horizon. The background metric in these calculations a priori should not satisfy any field equations. Thus, that the conical singularity method is an off-shell method is a clear advantage.

The applicability of the method became even wider after the formulation of the holographic description of entanglement entropy [5] (and the proofs in [6] and [7]). The holography has put in the focus the conformal field theories. In a related development it was studied a relation between the trace anomaly in a 4d CFT and the logarithmic terms in entanglement entropy. It was found in [8] that this is a one-to-one relation which, in particular, involves the extrinsic geometry of the entangling surface. This observation was the first indication of a departure from the Wald entropy. In terms of the distributional geometry of conical singularities it manifests in the appearance of the extrinsic curvature contribution in the integrals of the invariants quadratic in curvature, as was demonstrated in [9]. Building on these approaches some further generalizations for more general curvature invariants [10,11] and applications for holographic calculations [12–14] have appeared in the literature. On the other hand, there have not yet been much progress in understanding the entropy which originates from the invariants which involve derivatives of the curvature.

Among the numerous results obtained in the recent years that overwhelmingly confirmed the theoretical predictions there was one observation which has not yet found its place in the otherwise harmonic picture. This observation made by Hung, Myers and Smolkin in 2011 [12] concerns the entropy in $d=6$ conformal field theory. They have found that there is a discrepancy between the holographic calculation of entanglement entropy (using the Jacobson–Myers functional for the holographic minimal surface) and the CFT trace anomaly calculation if one uses the Wald prescription to compute the entropy. This discrepancy appears in six dimensions and it is apparently due to the B_3 conformal charge. In [12] there have been given four examples of rather simple six-dimensional spacetimes and four-dimensional entangling surfaces for which this discrepancy appears. In all these examples the

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http://dx.doi.org/10.1016/j.physletb.2015.10.036
0370-2693/

entangling surfaces are minimal surfaces so that the discrepancy cannot be due to the extrinsic curvature which vanishes in these examples. On the other hand, the $O(2)$ symmetry in the transverse subspace is not present for the surfaces considered in [12]. Thus, it was emphasized that the discrepancy might be due to the lack of this symmetry.

In our recent work [1] we have suggested that, in a more general context, the total derivative terms in the gravitational action might produce some non-trivial contributions to the entropy. This is yet a more radical deviation from the Wald prescription for the entropy. In particular, the total derivative terms in the trace anomaly may give rise to some important contributions to the logarithmic terms in entanglement entropy of a CFT. As an application of this general statement we have suggested that the discrepancy of Hung, Myers and Smolkin may originate from the total derivative terms present in the trace anomaly in six dimensions and have proposed some "phenomenological formula" for the discrepancy. In the present note we finalize this proposal and identify explicitly the total derivative term in the $d=6$ trace anomaly that is responsible for the discrepancy.

2. Regularized metrics and "phenomenological" method

Motivated by examples of [12] we shall consider the metric of the following general form

$$\begin{aligned} ds^2 &= e^{2\sigma(x,r)}[dr^2+r^2d\tau^2]+g_{ij}(x,r,\tau)dx^idx^j, \\ g_{ij}(x,r,\tau) &= h_{ij}(x)+\frac{1}{2}H_{ij}(x)r^2+\tilde{H}^{ab}_{ij}(x)n^an^br^2+\cdots, \\ \sigma(x,r) &= \sigma_0(x)+\frac{1}{2}\sigma_2(x)r^2+\cdots, \end{aligned} \tag{1}$$

where $H_{ij}=H^{ab}_{ij}(x)\delta_{ab}$ is the trace part and $\tilde{H}^{ab}_{ij}=H^{ab}_{ij}-\frac{1}{2}\delta_{ab}H_{ij}$ is the traceless part of H^{ab}, and $n^1=\cos(\tau)$, $n^2=\sin(\tau)$. In (1) we deliberately did not include the terms with extrinsic curvature as we want to study closely the case of [12]. The entangling surface Σ is at $r=0$ in metric (1). Applying the replica trick (for a review on this method see [15]) we change the periodicity of τ to be from 0 to $2\pi n$, where n is an integer. In fact in the replica method we continue to non-integer values of n. This introduces a conical singularity at $r=0$. In order to treat this singularity properly we have to regularize the metric.

Fursaev–Solodukhin (FS) regularization. One possible regularization is the one introduced in [4]. It consists in replacing the metric component g_{rr} (1) with $e^{2\sigma(x,r)}f_n(r)$, where

$$f_n(r)=\frac{r^2+b^2n^2}{r^2+b^2} \tag{2}$$

is the regularization function. At the end of the calculation we are supposed to take the limit $b\to 0$. In many known cases this regularization gives the Wald entropy. However, with this regularization alone the regularized metric is characterized by the curvature which is everywhere finite but its derivatives may diverge at $r=0$. Therefore, one should supplement it with some other regularization.

Generalized (G) regularization. This regularization is a generalization of the one introduced in [9]. It is based on the observation that the divergence in the gradients of the curvature is entirely due to the traceless term $\tilde{H}^{ab}_{ij}$ in the metric (1). Therefore, as suggested in [11], one has to regularize this part of the metric by replacing

$$H^{ab}_{ij}(x)n^an^br^2\to\frac{1}{2}H_{ij}(x)r^2+(H^{ab}_{ij}(x)-\frac{1}{2}\delta^{ab}H_{ij}(x))n^an^br^{2n} \tag{3}$$

in the metric (1). In oder to make the derivatives of curvature regular we assume that n is slightly larger than 1. If the traceless part of H^{ab} vanishes then metric (1) possesses the Killing symmetry and describes a Killing horizon at $r=0$. For this metric the Wald calculation of entropy is applicable and we do not expect any modifications of this calculation. This explains why we did not modify the power of r in front of $H_{ij}(x)$ in (3). We stress that regularization (3) should be used in addition to the regularization with the function $f_n(r)$ (2). This generalized regularization was applied in [1] to the analysis of the contribution of some total derivative terms to the entropy.

Consider now a curvature invariant $\mathcal{J}$ which may include any function of curvature and its derivatives. Comparing the integrals of $\mathcal{J}$ in these two regularizations we see that their difference should vanish provided the traceless part $\tilde{H}^{ab}_{ij}$ vanishes. Therefore this difference is a function of $\tilde{H}^{ab}_{ij}$ only. To leading order, when only quadratic combinations are taken into account we have that

$$\begin{aligned} &\left[\int_{\mathcal{M}_n}\mathcal{J}\right]_{FS}-\left[\int_{\mathcal{M}_n}\mathcal{J}\right]_G \\ &=(1-n)\int_\Sigma(\alpha\,\mathrm{Tr}\,\tilde{H}^{ab}\,\mathrm{Tr}\,\tilde{H}^{ab}+\beta\tilde{H}^{ab}_{ij}\tilde{H}^{ab,ij}), \end{aligned} \tag{4}$$

where $\mathrm{Tr}\,\tilde{H}^{ab}=h^{ij}\tilde{H}^{ab}_{ij}$.

Provided the FS regularization produces the Wald entropy the difference (4) gives the desired discrepancy. In the case considered in [12] the invariant $\mathcal{J}=\mathcal{A}$ is the $d=6$ trace anomaly. In a "phenomenological" approach taken in [1] one can determine the unknown constants α and β by making (4) consistent with the examples considered in [12]. In fact, only two examples of [12] are sufficient to fix these constants. The expression obtained in [1] reads

$$\begin{aligned} &\left[\int_{\mathcal{M}_n}\mathcal{A}\right]_{FS}-\left[\int_{\mathcal{M}_n}\mathcal{A}\right]_G \\ &=4\pi(1-n)B_3\int_\Sigma(\mathrm{Tr}\,\tilde{H}^{ab}\,\mathrm{Tr}\,\tilde{H}^{ab}-4\tilde{H}^{ab}_{ij}\tilde{H}^{ab,ij}), \end{aligned} \tag{5}$$

where B_3 is the conformal charge which corresponds to invariant I_3 in the conformal anomaly. It can be rewritten in terms of the doubly traceless tensor $\hat{H}^{ab}_{ij}=\tilde{H}^{ab}_{ij}-\frac{1}{4}h_{ij}\,\mathrm{Tr}\,\tilde{H}^{ab}$ as follows

$$\left[\int_{\mathcal{M}_n}\mathcal{A}\right]_{FS}-\left[\int_{\mathcal{M}_n}\mathcal{A}\right]_G=16\pi(n-1)B_3\int_\Sigma\hat{H}^{ab}_{ij}\hat{H}^{ab,ij}. \tag{6}$$

This formula is equivalent to the Hung–Myers–Smolkin expression (equation (5.35) in [12]) written in terms of the Weyl tensor. Here (and in [1]) we derive this formula in two steps: first, comparing the two regularization we conclude that the entropy difference is due to the traceless part of H^{ab} that allowed us to reduce the possible contributions to only two terms (4). In the second step, in order to fix the unknown constants α and β we have used the values for the entropy discrepancy provided by any two independent examples considered in [12].

3. Conformal invariants in six dimensions

In a generic conformal field theory in $d=6$ the trace anomaly, modulo the total derivatives, is a combination of four different terms