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Bekenstein's generalized second law of thermodynamics: The role of the hoop conjecture

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In memory of Prof. Jacob D. Bekenstein 1947–2015

ABSTRACT

Bekenstein's generalized second law (GSL) of thermodynamics asserts that the sum of black-hole entropy, $S_{BH} = Ac^3/4\hbar G$ (here A is the black-hole surface area), and the ordinary entropy of matter and radiation fields in the black-hole exterior region never decreases. We here re-analyze an intriguing gedanken experiment which was designed by Bekenstein to challenge the GSL. In this historical gedanken experiment an entropy-bearing box is lowered into a charged Reissner–Nordström black hole. For the GSL to work, the resulting increase in the black-hole surface area (entropy) must compensate for the loss of the box's entropy. We show that *if* the box can be lowered adiabatically all the way down to the black-hole horizon, as previously assumed in the literature, then for *near-extremal* black holes the resulting increase in black-hole surface-area (due to the assimilation of the box by the black hole) may become too small to compensate for the loss of the box's entropy. In order to resolve this apparent violation of the GSL, we here suggest to use a generalized version of the hoop conjecture. In particular, assuming that a physical system of mass M and electric charge Q forms a black hole if its circumference radius r_c is equal to (or smaller than) the corresponding Reissner–Nordström black-hole radius $r_{RN} = M + \sqrt{M^2 - Q^2}$, we prove that a new (and larger) horizon is already formed before the entropy-bearing box reaches the horizon of the original near-extremal black hole. This result, which seems to have been overlooked in previous analyzes of the composed black-hole-box system, ensures the validity of Bekenstein's GSL in this famous gedanken experiment.

1. Introduction

The legend says [1,2] that it all began with a cup of tea and two genius physicists, Professor John Archibald Wheeler and his young student Jacob David Bekenstein, who tried to figure out what happens to the second law of thermodynamics when the cup goes down a black hole.

In this gedanken experiment, the thermal entropy of the tea disappears behind the black-hole horizon. Hence, at first glance, it seems that the second law of thermodynamics, which states that entropy cannot decrease, is violated in this physical process. In particular, to external observers it seems that the entropy of the visible universe decreases as the (entropy-bearing) object disappears into the black hole.

It was while attempting to resolve this apparent paradox that Bekenstein came up with the bold idea to associate entropy with black holes – entropy as the measure of (missing) information about the black-hole internal state which is inaccessible to external observers [3]. In particular, the formal analogy between the second law of thermodynamics and Hawking's area theorem [4], which states that black-hole surface area cannot decrease [5], motivated Bekenstein to conjecture that the required black-hole entropy [6] is proportional to its surface area A [3]:

$$S_{BH} = \frac{k_B A}{4 l_P^2}. \tag{1}$$

The Planck length $l_P = \sqrt{\hbar G/c^3}$ was introduced into (1) by Wheeler on dimensional grounds [2,7], whereas the correct proportionality coefficient, 1/4, was later found by Hawking [8].

Using the conjectured proportionality (1) between black-hole entropy and horizon area, Bekenstein proposed a generalized version of the second law of thermodynamics [3]: *The sum of black-hole entropy, S_{BH}, and the ordinary entropy of matter and radiation fields in the black-hole exterior region, S, cannot decrease*. This conjecture

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therefore asserts that physical processes involving black holes are characterized by the relation

$$\Delta(S_{\rm BH}+S)\geq 0. \tag{2}$$

The generalized second law of thermodynamics (GSL) provides a unique relation between thermodynamics, gravitation, and quantum theory [9]. It therefore allows us a unique glimpse into the elusive theory of quantum gravity. It should be emphasized, however, that despite the general agreement that the GSL reflects a fundamental aspect of the quantum theory of gravity, there currently exists no general proof (that is, a proof which is based on the fundamental microscopic laws of quantum gravity) for the validity of this principle. It is therefore of physical interest to consider gedanken experiments in order to test the validity of the GSL in various physical situations.

2. Bekenstein's universal entropy bound

In order to challenge the GSL, Bekenstein [3,10] analyzed a gedanken experiment in which a finite-sized object with negligible self-gravity is assimilated into a black hole [11]. In particular, Bekenstein showed that the capture of a spherical body of proper mass μ and radius R by a black hole produces an unavoidable increase ΔA in the black-hole surface area, whose *minimal* value is given by the relation [3,12]

$$(\Delta A)_{\rm min}=8\pi\mu R. \tag{3}$$

Taking cognizance of Eqs. (1), (2), and (3), Bekenstein [3,10] conjectured the existence of a universal upper bound,

$$S\leq \frac{2\pi\mu R}{\hbar}, \tag{4}$$

on the entropy content of physical systems with negligible self-gravity [13–16]. In particular, as emphasized by Bekenstein [3,10], an entropy bound of the form (4) ensures that the generalized second law of thermodynamics (2) is respected in a physical process in which a spherical body with negligible self-gravity is captured by a black hole [17]. It is worth mentioning that Bekenstein and others [10,18–20] provided compelling evidence that the entropy bound (4) is respected in various physical systems in which gravity is negligible.

The main goal of the present paper is to highlight a non-trivial aspect of Bekenstein's famous gedanken experiment [3]. In particular, we shall challenge the GSL in a gedanken experiment in which an entropy-bearing spherical body is slowly lowered into a charged Reissner–Nordström black hole. We shall show below that *if* the body can be lowered adiabatically all the way down to the black-hole horizon, as previously assumed in the literature, then for *near-extremal* black holes the unavoidable increase in black-hole surface-area [see Eq. (21) below] may become too small to compensate for the loss of the body's entropy [21]. We shall further develop a possible resolution of this apparent paradox. In particular, we shall show that a generalized version of the hoop conjecture [22] may ensure the validity of Bekenstein's GSL in this type of gedanken experiments.

3. Challenging the generalized second law of thermodynamics

We consider an entropy-bearing box of proper radius R and rest mass μ which is lowered towards a Reissner–Nordström (RN) black hole of mass M and electric charge Q. The external gravitational field of the RN black-hole spacetime is described by the line element

$$ds^2=-\Big(1-\frac{2M}{r}+\frac{Q^2}{r^2}\Big)dt^2+\Big(1-\frac{2M}{r}+\frac{Q^2}{r^2}\Big)^{-1}dr^2 +r^2d\Omega^2. \tag{5}$$

The black-hole (outer and inner) horizons are located at

$$r_{\pm}=M\pm\sqrt{M^2-Q^2}. \tag{6}$$

The test-particle approximation imposes the constraints

$$\mu\ll R\ll M. \tag{7}$$

These strong inequalities imply that the lowered body (the entropy-bearing box) has negligible self-gravity and that it is much smaller than the geometric size of the black hole.

Our goal is to challenge the GSL in the most extreme situation. We shall therefore consider the case of an entropy-bearing body which is *slowly* lowered towards the black hole. As shown by Bekenstein [3], this strategy guarantees that the energy delivered to the black hole when it swallows the body is as small as possible [23]. The Bekenstein strategy of lowering the body adiabatically into the black hole also guarantees that, for given parameters of the body, the resulting increase in the surface area (entropy) of the black hole is minimized [3].

The red-shifted energy (energy-at-infinity) of a static body which is located at a radial coordinate r in the RN black-hole spacetime is given by [3]

$$\mathcal{E}(r)=\mu\sqrt{1-\frac{2M}{r}+\frac{Q^2}{r^2}}. \tag{8}$$

This energy can be expressed in terms of the proper distance l of the body's center of mass above the black-hole horizon. Using the relation [3]

$$l(r)=\int_{r_+}^{r}\sqrt{g_{rr}}dr, \tag{9}$$

and taking cognizance of (5), one finds the exact relation

$$l(r)=\sqrt{(r-r_+)(r-r_-)}+2M\ln\Big(\frac{\sqrt{r-r_+}+\sqrt{r-r_-}}{\sqrt{r_+-r_-}}\Big). \tag{10}$$

From (10) one finds

$$r(l)=r_+ +(r_+-r_-)\frac{l^2}{4r_+^2}[1+O(l^2/r_+^2)] \tag{11}$$

in the near-horizon $l\ll r_+$ region. Substituting (11) into (8), one finds [3]

$$\mathcal{E}(l)=\frac{\mu l(r_+-r_-)}{2r_+^2} \tag{12}$$

for the red-shifted energy of the box in the RN black-hole spacetime [24–27].

Suppose the entropy-bearing box is lowered slowly towards the black hole until its center of mass lies a proper distance l_0 (with $l_0\geq R$) above the horizon. The box is then released to fall freely into the black hole. The energy (energy-at-infinity) delivered to the black hole when it captures the body is given by $\mathcal{E}(l=l_0)$. The increase

$$\Delta M=\mathcal{E}(l_0)=\frac{\mu l_0(r_+-r_-)}{2r_+^2} \tag{13}$$

in the mass of the RN black hole results in a change [see Eq. (6)] [28]