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Small-angle scattering and quasiclassical approximation beyond leading order

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ABSTRACT

In the present paper we examine the accuracy of the quasiclassical approach on the example of small-angle electron elastic scattering. Using the quasiclassical approach, we derive the differential cross section and the Sherman function for arbitrary localized potential at high energy. These results are exact in the atomic charge number and correspond to the leading and the next-to-leading high-energy small-angle asymptotics for the scattering amplitude. Using the small-angle expansion of the exact amplitude of electron elastic scattering in the Coulomb field, we derive the cross section and the Sherman function with a relative accuracy θ^2 and θ^1, respectively (θ is the scattering angle). We show that the correction of relative order θ^2 to the cross section, as well as that of relative order θ^1 to the Sherman function, originates not only from the contribution of large angular momenta $l \gg 1$, but also from that of $l \sim 1$. This means that, in general, it is not possible to go beyond the accuracy of the next-to-leading quasiclassical approximation without taking into account the non-quasiclassical terms.

1. Introduction

In the high-energy QED processes in the atomic field, the characteristic angles θ between the momenta of final and initial particles are small. Therefore, the main contribution to the amplitudes of the processes is given by the large angular momenta $l \sim \varepsilon\rho \sim \varepsilon/\Delta \sim 1/\theta$, where ε, ρ, and Δ are the characteristic energy, impact parameter, and momentum transfer, respectively ($\hbar = c = 1$). The quasiclassical approach provides a systematic method to account for the contribution of large angular momenta. It was successfully used for the description of numerous processes such as charged particle bremsstrahlung, pair photoproduction, Delbrück scattering, photon splitting, and others [1–8]. The accurate description of such QED processes is important for the data analysis in modern detectors of elementary particles. The quasiclassical approach allows one to obtain the results for the amplitudes not only in the leading quasiclassical approximation but also with the first quasiclassical correction taken into account [9–14]. We stress the difference between the quasiclassical approximation and the eikonal approximation often used in the description of the high-energy processes (see, e.g., Ref. [15]). This difference was recognized already in Ref. [3] where it was shown that the Coulomb corrections to the cross section of e^+e^- pair photoproduction can be obtained within the quasiclassical approach but not within the eikonal approximation.

A natural question arises: how far can we advance in increasing accuracy within the quasiclassical framework? In this paper we examine this question by considering the process of high-energy small-angle scattering of polarized electrons in the atomic field. The general form of this cross section reads (see, e.g., Ref. [15])

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\frac{d\sigma_0}{d\Omega}\left[1 + S\,\boldsymbol{\xi}\cdot(\boldsymbol{\zeta}_1+\boldsymbol{\zeta}_2) + T^{ij}\zeta_1^i\zeta_2^j\right], \quad \boldsymbol{\xi} = \frac{\boldsymbol{p}\times\boldsymbol{q}}{|\boldsymbol{p}\times\boldsymbol{q}|}, \tag{1}$$

where $d\sigma_0/d\Omega$ is the differential cross section of unpolarized scattering, $\boldsymbol{p}$ and $\boldsymbol{q}$ are the initial and final electron momenta, respectively, $\boldsymbol{\zeta}_1$ is the polarization vector of the initial electron, $\boldsymbol{\zeta}_2$ is the detected polarization vector of the final electron, S is the so-called Sherman function, and T^{ij} is some tensor. In Section 2 we use the quasiclassical approach to derive the small-angle expansion of the cross section of electron elastic scattering in arbitrary localized potential. As for the unpolarized cross section $d\sigma_0/d\Omega$, its leading and subleading terms with respect to the scattering angle θ are known for a long time [16]. They can both be calculated within the quasiclassical framework. We show that the Sherman function S in the leading quasiclassical approximation is proportional to θ^2. We compare this result with that obtained by means of the expansion with respect to the parameter $Z\alpha$ [17–21] (Z is the nuclear charge

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number, $\alpha \approx 1/137$ is the fine structure constant). The leading in $Z\alpha$ contribution to the Sherman function is due to the interference between the first and second Born terms in the scattering amplitude. In contrast to the quasiclassical result (proportional to θ^2), it scales as θ^3 at small θ. There is no contradiction between these two results because the expansion of our quasiclassical result with respect to $Z\alpha$ starts with $(Z\alpha)^2$. Therefore, depending on the ratio $Z\alpha/\theta$, the dominant contribution to the Sherman function is given either by the leading quasiclassical approximation or by the interference of the first two terms of the Born expansion. One could imagine that the terms $O(\theta^3)$ in the function S can be ascribed to the next-to-leading quasiclassical correction and, therefore, they come from the contribution of large angular momenta. However, by considering the case of a pure Coulomb field, we show in Section 3 that the account for the angular momenta $l \sim 1$ is indispensable for these terms. Thus, we are driven to the conclusion that, in general, it is not possible to go beyond the accuracy of the next-to-leading quasiclassical approximation without taking into account the non-quasiclassical terms.

2. Scattering of polarized electrons in the quasiclassical approximation

It is shown in Ref. [22] that the wave function $\psi_{\boldsymbol p}(\boldsymbol r)$ in the arbitrary localized potential $V(r)$ can be written as

$$\psi_{\boldsymbol p}(\boldsymbol r) = [g_0(\boldsymbol r, \boldsymbol p) - \boldsymbol\alpha\cdot \boldsymbol g_1(\boldsymbol r, \boldsymbol p) - \boldsymbol\Sigma\cdot\boldsymbol g_2(\boldsymbol r, \boldsymbol p)]u_{\boldsymbol p}\,,$$
$$u_{\boldsymbol p} = \sqrt{\frac{\varepsilon+m}{2\varepsilon}}\begin{pmatrix}\phi\\ \dfrac{\boldsymbol\sigma\cdot\boldsymbol p}{\varepsilon+m}\phi\end{pmatrix}, \tag{2}$$

where ϕ is a spinor, $\boldsymbol\alpha = \gamma^0\boldsymbol\gamma$, $\boldsymbol\Sigma = \gamma^0\gamma^5\boldsymbol\gamma$, m is the electron mass, and $\boldsymbol\sigma$ are the Pauli matrices. In this section we assume that $m/\varepsilon \ll 1$. In the leading quasiclassical approximation, the explicit forms of the functions g_0 and $\boldsymbol g_1$, as well as the first quasiclassical correction to g_0, are obtained in Ref. [9]. The first quasiclassical correction to $\boldsymbol g_1$ and the leading contribution to $\boldsymbol g_2$ are derived in Ref. [14]. The asymptotic form of the function $\psi_{\boldsymbol p}(\boldsymbol r)$ at large distances r reads

$$\psi_{\boldsymbol p}(\boldsymbol r) \approx e^{i\boldsymbol p\cdot\boldsymbol r}u_{\boldsymbol p} + \frac{e^{ipr}}{r}[G_0 - \boldsymbol\alpha\cdot\boldsymbol G_1 - \boldsymbol\Sigma\cdot\boldsymbol G_2]u_{\boldsymbol p}\,. \tag{3}$$

The functions G_0, $\boldsymbol G_1$, and $\boldsymbol G_2$ can be easily obtained from the expressions for g_0, $\boldsymbol g_1$, and $\boldsymbol g_2$ in Ref. [14]:

$$G_0 = f_0 + \delta f_0\,,\quad \boldsymbol G_1 = -\frac{\boldsymbol\Delta_\perp}{2\varepsilon}[f_0+\delta f_0+\delta f_1],$$
$$\boldsymbol G_2 = i\frac{[\boldsymbol q\times\boldsymbol p]}{2\varepsilon^2}\delta f_1\,, \tag{4}$$

where

$$f_0 = -\frac{i\varepsilon}{2\pi}\int d\boldsymbol\rho\, e^{-i\boldsymbol\Delta_\perp\cdot\boldsymbol\rho}\left[e^{-i\chi(\boldsymbol\rho)}-1\right],$$
$$\delta f_0 = -\frac{1}{4\pi}\int d\boldsymbol\rho\, e^{-i\boldsymbol\Delta_\perp\cdot\boldsymbol\rho - i\chi(\boldsymbol\rho)}\rho\frac{\partial}{\partial\rho}\int_{-\infty}^{\infty}dx V^2(r_x)$$
$$\delta f_1 = \frac{i}{4\pi\Delta_\perp^2}\int d\boldsymbol\rho\, e^{-i\boldsymbol\Delta_\perp\cdot\boldsymbol\rho - i\chi(\boldsymbol\rho)}\boldsymbol\Delta_\perp\cdot\frac{\boldsymbol\rho}{\rho}\frac{\partial}{\partial\rho}\int_{-\infty}^{\infty}dx V^2(r_x)\,,$$
$$\chi(\boldsymbol\rho) = \int_{-\infty}^{\infty}dx V(r_x)\,,\quad r_x = \sqrt{x^2+\rho^2}\,. \tag{5}$$

Here $\boldsymbol\Delta = \boldsymbol q - \boldsymbol p$, $\boldsymbol q = p\boldsymbol r/r$, $\boldsymbol\rho$ is a two-dimensional vector perpendicular to the initial momentum $\boldsymbol p$, and the notation $\boldsymbol X_\perp = \boldsymbol X - (\boldsymbol X\cdot\boldsymbol n_p)\boldsymbol n_p$ is used for any vector $\boldsymbol X$, $\boldsymbol n_p = \boldsymbol p/p$. For small scattering angle $\theta \ll 1$, we have $\delta f_0 \sim \delta f_1 \sim \theta f_0$. Taking this relation into account, we obtain the following expressions for $\frac{d\sigma_0}{d\Omega}$, T^{ij}, and S in Eq. (1)

$$\frac{d\sigma_0}{d\Omega} = |f_0|^2\left[1 + 2\,\mathrm{Re}\,\frac{\delta f_0}{f_0}\right], \tag{6}$$
$$T^{ij} = \delta^{ij} + \theta\epsilon^{ijk}\xi^k\,,$$
$$S = -\frac{m\theta}{\varepsilon}\,\mathrm{Im}\,\frac{\delta f_1}{f_0}\,. \tag{7}$$

In Eqs. (6) and (7) we keep only the leading and the next-to-leading terms with respect to θ in $d\sigma_0/d\Omega$ and T^{ij}, and the leading term in the function S. The form of T^{ij} is a simple consequence of helicity conservation in ultrarelativistic scattering. The expression for $d\sigma_0/d\Omega$ coincides with that obtained in the eikonal approximation [16]. Note that $f_0 \to -f_0^*$, $\delta f_0 \to \delta f_0^*$, and $\delta f_1 \to \delta f_1^*$ at the replacement $V \to -V$ as it simply follows from Eq. (5). Therefore, the quasiclassical result for the Sherman function S, Eq. (7), is invariant with respect to the replacement $V \to -V$. In contrast, the term $2\,\mathrm{Re}(\delta f_0/f_0)$ in $d\sigma_0/d\Omega$ in Eq. (6) results in the charge asymmetry in scattering, i.e., in the difference between the scattering cross sections of electron and positron, see, e.g., Ref. [15]. Similarly, the account for the first quasiclassical correction leads to the charge asymmetry in lepton pair photoproduction and bremsstrahlung in an atomic field [13,14,22].

Let us specialize Eqs. (6) and (7) to the case of a Coulomb field. Substituting $V(r) = -Z\alpha/r$ in Eq. (5), we have

$$f_0 = \frac{2\eta}{\varepsilon\theta^{2-2i\eta}}\frac{\Gamma(1-i\eta)}{\Gamma(1+i\eta)}\,,$$
$$\frac{\delta f_0}{f_0} = \frac14\pi\theta\eta h(\eta)\,,\quad \frac{\delta f_1}{f_0} = -\frac{\pi\theta\eta h(\eta)}{4(1+2i\eta)}\,,$$
$$h(\eta) = \frac{\Gamma(1+i\eta)\Gamma(1/2-i\eta)}{\Gamma(1-i\eta)\Gamma(1/2+i\eta)}\,, \tag{8}$$

where $\eta = Z\alpha$ and $\Gamma(x)$ is the Euler Γ function. Then, from Eqs. (6) and (7) we obtain

$$\frac{d\sigma_0}{d\Omega} = \frac{4\eta^2}{\varepsilon^2\theta^4}\left[1 + \frac{\pi\theta\eta}{2}\,\mathrm{Re}\,h(\eta)\right], \tag{9}$$
$$S = \frac{\pi m\eta\theta^2}{4\varepsilon}\,\mathrm{Im}\,\frac{h(\eta)}{1+2i\eta}\,. \tag{10}$$

The remarkable observation concerning the obtained Sherman function (10) is that it scales as θ^2 while the celebrated Mott result [17] for the leading in η contribution to S scales as $\theta^3\ln\theta$. There is no contradiction because the expansion of (10) in η starts with η^2, while the Mott result is proportional to η. Thus, the Mott result is not applicable if $\theta \lesssim \eta$. In the next section we obtain the result (10), along with smaller corrections with respect to θ, by expanding the exact Coulomb scattering amplitude represented as a sum of partial waves. We show that the Mott result is recovered in the order θ^3, as it should be.

Let us now qualitatively discuss the influence of the finite nuclear size on the cross section $d\sigma_0/d\Omega$ and the Sherman function S. We use the model potential

$$V(r) = -\frac{\eta}{\sqrt{r^2+R^2}}\,, \tag{11}$$

where R is the characteristic nuclear size. For this potential we take all integrals in Eq. (4) and obtain