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New Hamiltonian constraint operator for loop quantum gravity

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ABSTRACT

A new symmetric Hamiltonian constraint operator is proposed for loop quantum gravity, which is well defined in the Hilbert space of diffeomorphism invariant states up to non-planar vertices with valence higher than three. It inherits the advantage of the original regularization method to create new vertices to the spin networks. The quantum algebra of this Hamiltonian is anomaly-free on shell, and there is less ambiguity in its construction in comparison with the original method. The regularization procedure for this Hamiltonian constraint operator can also be applied to the symmetric model of loop quantum cosmology, which leads to a new quantum dynamics of the cosmological model.

The singularity theorem of general relativity (GR) is a strong signal that the classical Einstein's equations cannot be trusted when the spacetime curvature grows unboundedly. It is widely expected that a quantum theory of gravity would overcome the singularity problem of classical GR. A very lesson that one can learn from GR is that the spacetime geometry itself becomes dynamical. To carry out this crucial idea raised by Einstein 100 years ago, loop quantum gravity (LQG) is notable for its nonpertuibative and background-independent construction [1–4]. The kinematical Hilbert space of LQG consists of cylindrical functions over finite graphs embedded in the spatial manifold. The quantum geometric operators corresponding to area [5,6], volume [5,7,8], length [9–11], ADM energy [12] and quasi-local energy [13], etc. have discrete spectrums. The LQG quantization framework can also be generalized to high-dimensional GR [14] and scalar-tensor theories of gravity [15,16]. A crucial topic now in LQG is its quantum dynamics, which is being attacked from both the canonical LQG and the path integral approach of spin foam models. In the canonical approach a suitable regularization procedure was first proposed by Thiemann to obtain well-defined Hamiltonian constraint operators [17]. The Hamiltonian constraint operators obtained in this way will attach new arcs (edges) and hence create new trivalent co-planar vertices to the graph of the cylindrical function upon which they act [17,18]. The quantum dynamics determined by the Hamiltonian constraint operator is well tested in the symmetric models of loop quantum cosmology (LQC) [19]. The classical big bang singularities are resolved by quantum bounces in the models [20–22]. However, there are ambiguities in the graph-dependent triangulation construction of this operator. There is no unique way to average over different choices of the triangulation. Moreover, in order to obtain the on shell anomaly-free quantum algebra of the Hamiltonian constraint operator [23], one has to employ degenerate triangulation at the co-planar vertices of spin networks in the regularization procedure of the Hamiltonian.[1] This treatment implies that the regularization procedure has essentially neglected the Hamiltonian at the co-planar vertices before acting the regulated operator on them. Otherwise, this kind of Hamiltonian constraint operator would generate an anomalous algebra in the full theory, unless one inputs certain unnatural requirement to the interaction manner of the edges of the graph and the arcs added by the Hamiltonian operator [24]. The Hamiltonian constraint operators proposed recently in [25–27] do not generate new vertices on the graph of the cylindrical function and hence are anomaly-free on shell. However this kind of action cannot match the quantum dynamics of spin foam models where new vertices are unavoidable in their construction [28]. A regularization of the Hamiltonian constraint compatible with the spinfoam dynamics was considered in [29]. However, the resulted Hamiltonian operator acts nontrivially on the vertices that it created and thus has still an anomalous quantum algebra. It is therefore natural to ask the question whether one can construct some Hamiltonian constraint operator with the following properties: (i) it is well defined in a suitable Hilbert space, symmetric and anomaly-free; (ii) it generates new

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vertices; (iii) its action on co-planar vertices is not neglected by some special regularization procedure, and there is no special restriction on the interaction manner of the edges of the graph and the arcs added by its action. We will show that the answer is affirmative. An alternative quantization of the Hamiltonian constraint in LQG possessing the above three properties will be proposed. The regularization procedure of the Hamiltonian operator can also be applied to LQC models.

The Hamiltonian formalism of GR is formulated on a 4-dimensional manifold $M=\mathbb{R}\times\Sigma$, with Σ being a 3-dimensional spatial manifold. In connection dynamics, the canonical variables on Σ are the $SU(2)$-connection A_a^i and the densitized triad $\tilde{E}^b_j$, with the only nontrivial Poisson bracket $\{A_a^i(x),\tilde{E}^b_j(y)\}=\kappa\beta\delta^3(x,y)$, where $\kappa\equiv 8\pi G$ and β is the Barbero–Immirzi parameter. The Hamiltonian constraint reads

$$H(N)=\frac{1}{2\kappa}\int_\Sigma d^3x\, N\frac{\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{\det(q)}}\left(\epsilon_{ijk}F^k_{ab}-2(1+\beta^2)K^i_{[a}K^j_{b]}\right)$$
$$=: H^E(N)-T(N), \tag{1}$$

where $F^i_{ab}\equiv 2\partial_{[a}A^i_{b]}+\epsilon^i{}_{jk}A^j_aA^k_b$ is the curvature of A^i_a, K^i_a is the extrinsic curvature of Σ, and $\det(q)$ is the determinate of 3-metric $q_{ab}\equiv e^i_ae^j_b\delta_{ij}$ with e^i_a being the co-triad. $H^E(N)$ and $T(N)$ are called the Euclidean and Lorentzian terms of the Hamiltonian constraints respectively. Both $H^E(N)$ and $T(N)$ depend on the canonical variables in non-polynomial ways. Besides the indication of spin foam models, it is argued in [30] that the momentum variables in $H^E(N)$ also imply the creation of new vertices by its action. Thus we adopt the so-called semi-quantized regularization approach developed in [31] to derive a new Hamiltonian constraint operator, which creates new vertices as well. The Hamiltonian is not neglected at the co-planar vertices of spin networks by the regularization. But the result of its action on the co-planar vertices is zero. Hence it has an anomaly-free algebra on shell.

Let us first consider $H^E(N)$. By introducing a characteristic function $\chi_\epsilon(x,y)$ such that $\lim_{\epsilon\to 0}\chi_\epsilon(x,y)/\epsilon^3=\delta^3(x,y)$ and using the point-splitting scheme, it can be regularized as

$$H^E(N)=\frac{1}{2\kappa}\lim_{\epsilon\to 0}\int_\Sigma d^3x\, N(x)V^{-1/2}_{(x,\epsilon)}\epsilon_{ijk}F^i_{ab}(x)\tilde{E}^a_j(x)$$
$$\times\int_\Sigma d^3y\,\chi_\epsilon(x,y)\tilde{E}^b_k(y)V^{-1/2}_{(y,\epsilon)}, \tag{2}$$

where $V_{(x,\epsilon)}:=\epsilon^3\sqrt{\det(q)}(x)$. Since the volume operator has a large kernel, the naive inverse volume operator is not well defined. However, one can use the idea in [32] to circumvent this problem by defining a permissible inverse square root of volume operator as

$$\widehat{V^{-1/2}_{(y,\epsilon)}}:=\lim_{\lambda\to 0}(\hat{V}_{(y,\epsilon)}+\lambda\ell_{\rm p}^3)^{-1}\hat{V}^{1/2}_{(y,\epsilon)}, \tag{3}$$

where $\hat{V}_{(y,\epsilon)}$ is the *standard* volume operator in LQG (see [7]) corresponding to the volume of the cube with center y and radial ϵ. It is easy to see that qualitatively $\widehat{V^{-1/2}}$ has the same properties as $\hat{V}$. Thus we can promote the classical volume in (2) into its quantum version (3) and replace both densitized triads in (2) by corresponding operators $\hat{E}^b_k(y)=-i\beta\ell_{\rm p}^2\delta/\delta A^k_b(y)$ where $\ell_{\rm p}^2=\hbar\kappa$. Acting on a cylindrical function f_γ, the result formally reads

$$\frac{(-i\beta\ell_{\rm p}^2)^2}{2\kappa}\lim_{\epsilon\to 0}\int_0^1 dt'\int_0^1 dt\left\{\sum_{e'\neq e}\chi_\epsilon\left(e'(t'),e(t)\right)N(e'(t'))V^{-1/2}_{(e'(t'),\epsilon)}\right.$$
$$\times\left[\epsilon_{ijk}F^i_{ab}(e'(t'))\dot{e}'^a(t')\dot{e}^b(t)\right]X^j_{e'}(t')X^k_e(t)\widehat{V^{-1/2}_{(e(t),\epsilon)}}$$
$$+\sum_e\chi_\epsilon\left(e(t'),e(t)\right)N(e(t'))V^{-1/2}_{(e(t'),\epsilon)}\left[\epsilon_{ijk}F^i_{ab}(e(t'))\dot{e}^a(t')\dot{e}^b(t)\right]$$
$$\left.\times\left[\theta(t,t')X^{kj}_e(t,t')+\theta(t',t)X^{jk}_e(t',t)\right]\widehat{V^{-1/2}_{(e(t),\epsilon)}}\right\}\cdot f_\gamma, \tag{4}$$

where $\theta(t,t')=1$ for $t'>t$ and zero otherwise, $X^k_e(t):=\mathrm{tr}[(h_{e(0,t)}\tau_kh_{e(t,1)})^T\partial/\partial h_{e(0,1)}]$, $X^{jk}_e(t',t):=\mathrm{tr}[(h_{e(0,t')}\tau_jh_{e(t',t)}\tau_kh_{e(t,1)})^T\partial/\partial h_{e(0,1)}]$, here $\tau_k:=-\frac{i}{2}\sigma_k$ with σ_k being the Pauli matrices, and T denotes transpose. Partitioning of the domain $[0,1]$ as N segments by inputing $N-1$ points, $0=t_0,t_1,\cdots,t_{N-1},t_N=1$, and setting $\Delta t_n\equiv t_n-t_{n-1}\equiv\delta$, the integral in (4) can be replaced by the Riemann's sum in a limit. Then (4) reduces to

$$\frac{(-i\beta\ell_{\rm p}^2)^2}{2\kappa}\lim_{\delta\to 0}\lim_{\epsilon\to 0}\delta^2\left\{\sum_{e'\neq e}\sum_{n,m=1}^N\chi_\epsilon\left(e'(t'_{m-1}),e(t_{n-1})\right)\right.$$
$$\times N(e'(t'_{m-1}))V^{-1/2}_{\left(e'(t'_{m-1}),\epsilon\right)}$$
$$\times\left[\epsilon_{ijk}F^i_{ab}(e'(t'_{m-1}))\dot{e}'^a(t'_{m-1})\dot{e}^b(t_{n-1})\right]$$
$$\times X^j_{e'}(t'_{m-1})X^k_e(t_{n-1})\widehat{V^{-1/2}_{(e(t_{n-1}),\epsilon)}}$$
$$+\sum_e\sum_{n,m=1}^N\chi_\epsilon\left(e(t_{m-1}),e(t_{n-1})\right)N(e(t_{m-1}))V^{-1/2}_{(e(t_{m-1}),\epsilon)}$$
$$\times\left[\epsilon_{ijk}F^i_{ab}(e(t_{m-1}))\dot{e}^a(t_{m-1})\dot{e}^b(t_{n-1})\right]$$
$$\times\left[\theta(t_{n-1},t_{m-1})X^{kj}_e(t_{n-1},t_{m-1})\right.$$
$$\left.\left.+\theta(t_{m-1},t_{n-1})X^{jk}_e(t_{m-1},t_{n-1})\right]\widehat{V^{-1/2}_{(e(t_{n-1}),\epsilon)}}\right\}\cdot f_\gamma. \tag{5}$$

Since the volume operator and hence $\widehat{V^{-1/2}_{(y,\epsilon)}}$ vanish at divalent vertices, for sufficiently small ϵ, the only non-vanishing terms of the summation in (5) correspond to those of $m=n=1$. Moreover, the terms corresponding to $m=n=1$ in the second summation of (5), which involves only summation over $e(=e')$, vanish due to $\epsilon_{ijk}F^i_{ab}(e(t_0))\dot{e}^a(t_0)\dot{e}^b(t_0)=0$. For $e\neq e'$, we have

$$\delta^2F^i_{ab}(e'(0))\dot{e}'^a(0)\dot{e}^b(0)\approx\frac{2}{\mathcal{N}_\ell}\mathrm{sgn}(e',e)\mathrm{tr}_\ell\left(h_{\alpha_{e'e}}\tau_i\right), \tag{6}$$

where $\mathcal{N}_\ell=-\frac{\ell(\ell+1)(2\ell+1)}{3}$ with ℓ a half-integer representing a spin representation of $SU(2)$, $\alpha_{e'e}$ is a loop formed by adding an arc between $e'(\delta)$ and $e(\delta)$, and $\mathrm{sgn}(e',e):=\mathrm{sgn}[\epsilon_{ab}\dot{e}'^a(0)\dot{e}^b(0)]$ is the orientation factor which can be promoted into its quantum operator. From the property of $\hat{V}$, we know that $\widehat{V^{1/2}_{(v,\epsilon)}}\equiv\widehat{V^{1/2}_v}$ is independent of ϵ. Thus we can take the trivial limit $\epsilon\to 0$ and obtain

$$\hat{H}^E_\delta(N)\cdot f_\gamma:=\frac{(\beta\ell_{\rm p}^2)^2}{\kappa\mathcal{N}_\ell}\sum_{v\in V(\gamma)}N_v\widehat{V^{-1/2}_v}$$
$$\times\left[\sum_{e\cap e'=v}\mathrm{sgn}(e',e)\epsilon_{ijk}\mathrm{tr}_\ell\left(h_{\alpha_{e'e}}\tau_i\right)J^j_{e'}J^k_e\right]\widehat{V^{-1/2}_v}\cdot f_\gamma$$
$$=:\sum_{v\in V(\gamma)}N_v\sum_{e\cap e'=v}\hat{H}^E_{v,e'e} \tag{7}$$