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Inflation on a non-commutative space–time

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ABSTRACT

We study inflation on a non-commutative space–time within the framework of enveloping algebra approach which allows for a consistent formulation of general relativity and of the standard model of particle physics. We show that within this framework, the effects of the non-commutativity of spacetime are very subtle. The dominant effect comes from contributions to the process of structure formation. We describe the bound relevant to this class of non-commutative theories and derive the tightest bound to date of the value of the non-commutative scale within this framework. Assuming that inflation took place, we get a model independent bound on the scale of space–time non-commutativity of the order of 19 TeV.

1. Introduction

The idea of space–time non-commutativity dates back to the early days of quantum field theory when it was hoped that it may help to make sense of UV divergences which are intrinsic to this framework [1,2]. With the advent of renormalization and the proof that physically relevant Yang–Mills theories were renormalizable, non-commutative gauge theories lost much of their appeal. However, there was a renewal of interest for such theories when they reappeared as a certain limit in string theory [3,4]. In [4], it was shown that the end points of open strings ending on a *Dp*-brane with a Neveu–Schwarz two form flux B background do not commute. String theory has an additional symmetry transformation known as T-duality, which relates geometric structures in different topologies. It naturally gives rise to non-commutative geometry. Independently of string theory, quantum gravity is likely to involve the notion of a minimal length, see e.g. [5,6], which could imply a non-commutativity of space–time at short distances. This may help to alleviate the problem of the non-renormalizability of perturbative quantum gravity.

There are different approaches to non-commutative geometry, which can be divided in roughly two classes. The first approach is due to Alain Connes. It is based on the notion of the spectral triple and has its origin in mathematical physics. The second approach indeed goes back to Moyal and Groenewold [1,2] and emphasizes that space–time itself might be non-commutativity at short distance. The non-commutativity of space–time leads to issues with space–time and gauge symmetries. There are two distinct ways to deal with these issues. One is to take gauge fields to be as usual Lie algebra valued and to restrict the gauge symmetries which can be considered (see e.g. [4]). The other one is to take gauge fields in the enveloping algebra which enables one to consider any gauge group with any representation for the matter fields [7–11]. In this article, we will consider the latter approach and derive the tightest bound to date on the non-commutative scale within this approach.

We shall focus here on the simplest model of space–time non-commutativity which has been extensively studied and will consider non-commuting coordinates with a canonical structure

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad (1)$$

where $\theta^{\mu\nu}$ is a constant tensor of mass dimension -2.

Our aim is to investigate effects of space–time non-commutativity in the early universe. We thus have to select a framework which enables us to formulate both field theories and general relativity on a non-commutative space–time. While there are different approaches to space–time non-commutativity, there is only one which leads to the well-known standard model of particle physics and general relativity in the low energy regime. We shall thus use the enveloping algebra approach [7–11] which enables one to formulate any gauge theory including arbitrary representations for the gauge and matter fields on a non-commutative space–time. This approach has led to a consistent formulation of the standard model of particle physics on such a space–time [12]. Treating General Relativity as a gauge theory, one can also formulate General Relativity on a non-commutative space–time [13–15]. It turns

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out that one needs to limit general coordinate transformations to those which are volume preserving diffeomorphisms. This leads to unimodular gravity which is known to be, at least classically, equivalent to general relativity. Following the enveloping algebra approach has several benefits. First of all, it makes use of real symmetries which imply a conserved charged via Noether's theorem. Such theories have an exact space–time symmetry [16,17] which corresponds to Lorentz invariance in the limit of $\theta^{\mu\nu} \to 0$. The implication of this symmetry is that all the bounds on space–time non-commutativity are weak [18], typically of the order of a TeV [19,20].

Using this framework, we will consider inflation and the cosmic microwave background on a non-commutative space–time. There are many attempts to study inflation in the context of a non-commutative space–time [21–29],[1] but as far as we know this is the first study of early universe physics using the enveloping algebra approach which allows to study in details the effects of the non-commutativity of space–time on the metric. As an example we will consider chaotic inflation [30] on a non-commutative space–time and show that the effects of non-commutativity vanish both for the scalar field and for the metric. This is a rather surprising and interesting result since one might have expected that a preferred direction in space–time could lead to large effects in the slow role parameters since inflation could have exponentially increased the original asymmetry in space–time. We then consider the effects of space–time non-commutativity on the CMB which are this time non-vanishing. This is not surprising as non-commutative gauge theories are a special case of non-local theories which are known to affect the CMB. We derive the tightest bound to date on the scale of space–time non-commutativity within this framework.

2. Theoretical framework

We consider here the algebra $\hat{\mathcal{A}}$ of non-commutative space–time coordinates $\{\hat{x}^\mu\}$ satisfying the canonical relation

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{2}$$

where $\theta \in \Omega^2(T\mathcal{M})$ is a constant tensor and can be locally expressed as $\theta = \theta^{\mu\nu}\partial_\mu \otimes \partial_\nu$ with $\theta^{\mu\nu} = -\theta^{\nu\mu}$. As usual, we want to represent functions in $\hat{\mathcal{A}}$ as elements in the space of linear complex functions $\mathcal{F}$. To do so we introduce the Moyal star product

$$\begin{aligned}(f_1 \cdot f_2)(\hat{x}) &= (f_1 \star f_2)(x)\\ &= \sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n \frac{1}{n!} \theta^{\mu_1\nu_1} \cdots \theta^{\mu_n\nu_n} \partial_{\mu_1} \cdots \partial_{\mu_n} f_1 \partial_{\nu_1} \cdots \partial_{\nu_n} f_2.\end{aligned} \tag{3}$$

Before continuing on to the main discussion, it will be useful to note some useful properties of the star product. Firstly, under complex conjugation one has

$$(f_1 \star f_2)^* = f_2^* \star f_1^*. \tag{4}$$

Secondly, the trace property under integration implies that

$$\int d^4x (f_1 \star f_2)(x) = \int d^4x (f_1 \cdot f_2)(x) \tag{5}$$

and more generally, one also has the cyclicity property

$$\begin{aligned}&\int d^4x (f_1 \star \cdots \star f_n)(x)\\ &= \int d^4x (f_1 \star \cdots \star f_{m-1}) \cdot (f_m \star \cdots \star f_n)(x)\\ &= \int d^4x (f_m \star \cdots \star f_n) \cdot (f_1 \star \cdots \star f_{m-1})(x).\end{aligned} \tag{6}$$

It is important to note, given that θ is constant, that this theory violates general diffeomorphism invariance. However, as shown in [13] we may recover a reduced group of diffeomorphisms compatible with (2) parametrized by

$$\hat{x}'^\mu = \hat{x}^\mu + \hat{\xi}^\mu. \tag{7}$$

A subset of these transformations given by

$$\hat{\xi}^\mu = \theta^{\mu\nu}\partial_\nu \hat{f}(\hat{x}) \tag{8}$$

leaves $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$ invariant. We shall thus only consider such transformations. Note that the Jacobian of these transformations is equal to one. The transformations which preserve the non-commutative algebra correspond to the reduced group of diffeomorphisms which are volume preserving. In other words, on a non-commutative space–time, we are forced to consider unimodular gravity. This is the main difference between our work and precious attempts at formulating inflation on a non-commutative space–time [22,25,26,31]. The approach to general relativity on a non-commutative space–time formulated in [13] relies on gauging a local SO(3, 1) (the tetrad approach). The local SO(3, 1) gauge symmetry is implemented using the enveloping algebra approach. This means that the gauge fields are assumed to be in the enveloping algebra instead of the usual Lie algebra. The local gauge invariance is enforced using the Seiberg–Witten maps order by order in θ [13]. We now have all the tools needed to formulate a consistent scalar field action in a curved space–time on a non-commutative space–time.

3. Non-commutative scalar action

We consider inflation driven a single scalar field with a potential $V(\phi^n)$ and denote for convenience $\phi \equiv \phi(x)$. In the commutative case, the action may be written

$$S = \int d^4x e \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \sum_n c_n \frac{\phi^n}{\Lambda^{(n-4)}}\right), \tag{9}$$

where e is the tetrad determinant, Λ is an energy scale and c_n are dimensionless Wilson coefficients of order unity. The choice of this frame follows from the derivation of non-commutative general relativity from the Seiberg–Witten map, as in [13,14], for which gravity is treated as a gauge theory. Another reason is that when mapping quantities on to a non-commutative space, it is very difficult to do so for a square root (which may not even exist in $\mathcal{A}_\theta$) and e is used as an effective way to represent $\sqrt{g}$. Setting the tetrad determinant to one, the action for the non-commutative scalar field may be written

$$\begin{aligned}S = \int d^4x \Bigg(&\frac{1}{2}G^{\mu\nu} \star \partial_\mu\phi \star \partial_\nu\phi\\ &- \frac{1}{2}m^2\phi \star \phi - \sum_n c_n \frac{\phi^{n\star}}{\Lambda^{(n-4)}}\Bigg).\end{aligned} \tag{10}$$

One might be tempted to take $\partial_\mu\phi \star \partial^\mu\phi$ and use (5) to eliminate the star product, as is done with, e.g., the mass term.

[1] These previous studies have mainly focussed on a non-commutative inflaton without considering non-commutative effects in the gravity sector. They have obtained bounds of the order of 10 TeV.