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Regularization methods for Nuclear Lattice Effective Field Theory

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ABSTRACT

We investigate Nuclear Lattice Effective Field Theory for the two-body system for several lattice spacings at lowest order in the pionless as well as in the pionful theory. We discuss issues of regularizations and predictions for the effective range expansion. In the pionless case, a simple Gaussian smearing allows to demonstrate lattice spacing independence over a wide range of lattice spacings. We show that regularization methods known from the continuum formulation are necessary as well as feasible for the pionful approach.

1. Introduction

The Nuclear Lattice Effective Field Theory (EFT) method [1] has led to impressive progress in the last decade and it has been applied to few- and many-body-systems successfully, for reviews see e.g. Refs. [2,3]. The lattice spacing serves as a natural UV regulator for the theory, as for a given value of a the maximal momentum is $p_{\max} = \pi/a$. Although these calculations give a quite good description for the phase shifts, energy levels, etc., almost all calculations have been done for a fixed lattice spacing $a \simeq 2$ fm, corresponding to a soft momentum cut-off of about 300 MeV. This allows one to treat all corrections beyond leading order (LO) in perturbation theory. However, the cut-off dependence or lattice spacing dependence has not been analyzed systematically and there are still some problems in the two-nucleon system like the relatively poor description of the $^3S_1$–$^3D_1$ mixing angle [4]. Further, such soft potentials seem to lead to some overbinding in medium-mass nuclei, as discussed in Ref. [5]. Also, it has been shown that the leading order four-nucleon contact interactions need to be smeared to avoid a cluster instability when four nucleons reside on one lattice site [6]. One might argue that the extension of such smearing methods also to the pion exchange contributions leads to a natural regularization of the lattice EFT, allowing to vary the lattice spacing freely but using an explicit momentum cut-off in the spirit of the work of Ref. [7]. More precisely, this inherent physical cut-off was implemented by formulating the lattice action in terms of blocked fields.

In this paper, we will focus on the neutron–proton two-body system at lowest order and discuss the lattice spacing dependence systematically. In addition, we discuss the necessity of regularizing the one-pion-exchange potential and provide a method that goes beyond smearing and is borrowed from continuum calculations, which leads to the lattice spacing independence of observables for a broad range in a, see Ref. [8].

While most of the calculations solve the transfer matrix using Monte Carlo methods or the Lanczos method for small eigenvalues of large sparse matrices, we use here the Hamiltonian formalism and solve it with the Lanczos method. Using this approach we can eliminate the discretization in the time direction and we have to consider only the variation in the position space discretization. In the following, all expressions are given in lattice units and one has to multiply the lattice results by the appropriate power of the lattice spacing a to get the physical values. Note also that we show simulations for various large enough volumes so that Lüscher's finite volume formulas are sufficient for the infinite volume extraction and we can entirely focus on the remaining dependence on the lattice spacing.

In what follows, we will first display the necessary formalism to calculate the neutron–proton system to lowest order on the lattice. It is important to already improve the free Hamiltonian so as to be as close as possible to the free non-relativistic dispersion relation. At very low energies, one can consider the theory with contact interactions only, the so-called pionless theory. As we will show, the smearing of the contact interactions can be used

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Table 1
Coefficients for the lattice discretization of the Laplacian, the dispersion relation and momentum components depending on the stretching factor N.

	$\mathcal{O}(a^4)$		$\mathcal{O}(a^2)$
ω_0	$N \cdot \frac{1}{9} + \frac{49}{36}$	o_0	0
ω_1	$N \cdot \frac{1}{6} + \frac{3}{2}$	o_1	$\frac{4}{3}$
ω_2	$N \cdot \frac{1}{15} + \frac{3}{20}$	o_2	$\frac{1}{6}$
ω_3	$N \cdot \frac{1}{90} + \frac{1}{90}$	o_3	0

as a regulator, leading to regulator-independent results for a broad range of values of the lattice spacing a. Matters are different in the pionful theory, which to LO consists of two four-nucleon contact interaction and the long-ranged static one-pion-exchange potential (OPEP). As will be shown, combining the smearing of the contact interactions with a position-space regularization of the OPEP will again lead to results largely independent of a for the physically sensible range of lattice spacing. Hence, one could use this modified leading-order approach to improve the current auxiliary field Monte Carlo simulations in Nuclear Lattice EFT. In principle, now it is possible to consider the continuum limit $a \to 0$, however, we refrain from doing that here, as it is sufficient to demonstrate lattice spacing independence for a physically relevant range of a.

2. The lattice Hamiltonian

To set the stage and to introduce our notations, we first discuss the free Hamiltonian. Its discretized form reads

$$H_{\text{free}} = \frac{1}{2m_N} \sum_{\boldsymbol{n},i,j} \sum_{\hat{\boldsymbol{l}}} \Bigg\{ 2\omega_0 a^\dagger_{i,j}(\boldsymbol{n})\, a_{i,j}(\boldsymbol{n}) + \sum_{k=1}^{3} (-1)^k \times \omega_k \left[a^\dagger_{i,j}(\boldsymbol{n})\, a_{i,j}\left(\boldsymbol{n} + k\hat{\boldsymbol{l}}\right) + a^\dagger_{i,j}(\boldsymbol{n})\, a_{i,j}\left(\boldsymbol{n} - k\hat{\boldsymbol{l}}\right) \right] \Bigg\}. \quad (1)$$

Here, $a_{i,j}$, $a^\dagger_{i,j}$ are the fermionic annihilation and creation operators with spin and isospin indices i, j, respectively, $m_N = (m_p + m_n)/2$ denotes the nucleon mass and $\hat{\boldsymbol{l}}$ is a unit vector in spatial direction. The summation is over all lattice points $\boldsymbol{n}$ on the L^3 lattice. We use a stretched $\mathcal{O}(a^m)$-improved action and its coefficients ω_k are summarized in Table 1, see e.g. Refs. [9,10]. m indicates the number of hopping points beyond next-neighbor interaction used in the Laplacian discretization in each spatial direction and we use $m = 4$ throughout this paper. The stretching factor N is introduced to minimize the errors arising from the discretized dispersion relations on the lattice especially for large momenta where the discretization does not approximate the continuum relation $E = \boldsymbol{p}^2/(2m_N)$ anymore. While there is some arbitrariness on the exact choice of N depending on the values of the respective momentum, $N = 3.5$ is a sensible choice.

The interaction potential consists of two/three terms in the pionless/pionful theory at lowest order. The contact interaction consists of two terms which can be chosen as

$$H_{\text{cont}} = \frac{1}{2} \sum_{\boldsymbol{n}} \Bigg[C \rho^{a^\dagger,a}(\boldsymbol{n})\, \rho^{a^\dagger,a}(\boldsymbol{n}) + C_I \sum_I \rho^{a^\dagger,a}_I(\boldsymbol{n})\, \rho^{a^\dagger,a}_I(\boldsymbol{n}) \Bigg], \quad (2)$$

where the terms are summed over all lattice sites $\boldsymbol{n}$ and the isospin index $I = 1, 2, 3$. These terms appear in both versions of the EFT considered here. In the pionful theory, one has in addition the one-pion-exchange potential

$$H_{\text{OPE}} = -\frac{g_A^2}{8F_\pi^2} \sum_{S_1,S_2,I} \sum_{\boldsymbol{n}_1,\boldsymbol{n}_2} G_{S_1S_2}(\boldsymbol{n}_1 - \boldsymbol{n}_2) \times \rho^{a^\dagger,a}_{S_1,I}(\boldsymbol{n}_1)\, \rho^{a^\dagger,a}_{S_2,I}(\boldsymbol{n}_2), \quad (3)$$

with g_A the axial-vector coupling constant and F_π the pion decay constant. S_1, S_2 are the respective spin indices which run from 1 to 3. The corresponding lattice density operators read

$$\rho^{a^\dagger,a}(\boldsymbol{n}) = a^\dagger_{i,j}(\boldsymbol{n})\, a_{i,j}(\boldsymbol{n}), \quad (4)$$

$$\rho^{a^\dagger,a}_I(\boldsymbol{n}) = a^\dagger_{i,j}(\boldsymbol{n})\, \tau_{I,jj'} a_{i,j}(\boldsymbol{n}), \quad (5)$$

$$\rho^{a^\dagger,a}_{S,I}(\boldsymbol{n}) = a^\dagger_{i,j}(\boldsymbol{n})\, \sigma_{S,ii'} \tau_{I,jj'} a_{i,j}(\boldsymbol{n}), \quad (6)$$

and $G_{S_1S_2}(\boldsymbol{n})$ represents the pion propagator times the pion–nucleon vertex and is defined as

$$G_{S_1S_2}(\boldsymbol{n}) = \frac{1}{L^3} \sum_{\boldsymbol{p}} \frac{\exp(-i\boldsymbol{p}\cdot\boldsymbol{n})\, \nu\left(p_{S_1}\right) \nu\left(p_{S_2}\right)}{1 + \frac{2}{q_\pi} \sum_{k=1}^{3} \sum_l (-1)^k \cos(kp_l)} \quad (7)$$

with $q_\pi = M_\pi^2 + 6\omega_0$. $\nu(p_{S_1})$, $\nu(p_{S_2})$ are the discretized momentum components of the first and second pion field which yields $\nu(p_l) = o_1 \sin(p_l) - o_2 \sin(2p_l) = p_l(1 + \mathcal{O}(p_l^4))$ with the coefficients summarized in Table 1. We only use an $\mathcal{O}(a^2)$ discretization, because we do not want to expand the respective interaction too much. A further improved momentum approximation is linked to a further expanded derivative in position space including more interactions at distinct lattice points and the locality of the pion–nucleon interaction is lost. These momenta arise from the pion field derivative in the pion nucleon Lagrangian $\mathcal{L}_{\pi N} = -g_A/(2F_\pi) N^\dagger \boldsymbol{\tau} \cdot (\boldsymbol{\sigma} \cdot \nabla) \boldsymbol{\pi} N$. To arrive at Eq. (7), we note that the pion propagator is derived from the discrete action for instantaneous pions which takes the form [6]

$$S_{\pi\pi}(\pi_I) = \left(\frac{m_\pi^2}{2} + 3\omega_0 \right) \sum_{\boldsymbol{n}} \pi_I(\boldsymbol{n})\, \pi_I(\boldsymbol{n}) + \sum_{\boldsymbol{n},\hat{\boldsymbol{l}},k} (-1)^k \omega_k \pi_I(\boldsymbol{n})\, \pi_I\left(\boldsymbol{n} + k\hat{\boldsymbol{l}}\right). \quad (8)$$

This is reparametrized by $\pi'_I(\boldsymbol{n}) = \sqrt{q_\pi}\pi_I(\boldsymbol{n})$. Finally, the new pion action reads

$$S_{\pi\pi}\left(\pi'_I\right) = \frac{1}{2} \sum_{\boldsymbol{n}} \pi'_I(\boldsymbol{n})\, \pi'_I(\boldsymbol{n}) + \frac{1}{q_\pi} \sum_{\boldsymbol{n},\hat{\boldsymbol{l}},k} (-1)^k \omega_k \pi'(\boldsymbol{n})\, \pi'_I\left(\boldsymbol{n} + k\hat{\boldsymbol{l}}\right) \quad (9)$$

and the respective pion propagator reads

$$D_\pi(\boldsymbol{p}) = \left[1 + \frac{2}{q_\pi} \sum_{k=1}^{3} \sum_l (-1)^k \cos(kp_l) \right]^{-1}. \quad (10)$$

Furthermore, we introduce a Gaussian smearing

$$f(\boldsymbol{p}) = \frac{1}{f_0} \exp\left[-b \frac{\tilde{\nu}(\boldsymbol{p})}{2} \right] \quad (11)$$