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Generalized dilatation operator method for non-relativistic holography

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ABSTRACT

We present a general algorithm for constructing the holographic dictionary for Lifshitz and hyperscaling violating Lifshitz backgrounds for any value of the dynamical exponent z and any value of the hyperscaling violation parameter θ compatible with the null energy condition. The objective of the algorithm is the construction of the general asymptotic solution of the radial Hamilton–Jacobi equation subject to the desired boundary conditions, from which the full dictionary can be subsequently derived. Contrary to the relativistic case, we find that a fully covariant construction of the asymptotic solution for running non-relativistic theories necessitates an expansion in the eigenfunctions of two commuting operators instead of one. This provides a covariant but non-relativistic grading of the expansion, according to the number of time derivatives.

1. Introduction

In recent years, great effort has been devoted to the use of holographic models in order to gain a deeper understanding of the strong coupling physics in condensed matter systems. The gauge/gravity duality has proven an instrumental tool in studying the strongly coupled dynamics near quantum critical points exhibiting Lifshitz [1,2] or Schrödinger [3,4] symmetry. More recently, gravity duals to non-relativistic systems that transform non-trivially under scale transformations have been put forward [5–8]. The geometries dual to such hyperscaling violating Lifshitz (hvLf) quantum systems are of the form

$$ds^2_{d+2} = \frac{du^2 - u^{-2(z-1)}dt^2 + d\vec{x}^2}{\ell^{-2}u^{2(d-\theta)/d}}, \tag{1}$$

where d is the spatial dimension, z and θ are respectively the Lifshitz and hyperscaling violation exponents, and ℓ is the Lifshitz radius. This metric transforms non-trivially under scale transformations as

$$\vec{x} \to \lambda\vec{x}, \qquad t \to \lambda^z t, \qquad u \to \lambda u, \qquad ds^2_{d+2} \to \lambda^{\frac{2\theta}{d}} ds^2_{d+2}. \tag{2}$$

By computing the energy of supergravity fluctuations around the background (1) one can unambiguously determine the location of the ultraviolet (UV) of the dual quantum field theory, corresponding to the conformal boundary of the geometry (1), to be at $u \to 0$, independently of the value of the exponents z and θ [7,9]. The only restriction we shall impose on the exponents z and θ is the null energy condition, which leads to seven distinct cases for the values of z and θ [10]. However, the only two solutions that allow for $z < 1$ require $\theta > d + z$, in which case the on-shell action is UV finite and as a result there are no well defined Fefferman–Graham asymptotic expansions [10]. The marginal case $\theta = d + z$ requires separate analysis. Our discussion here and in [10] therefore focuses on the case $z > 1$.

For earlier work on asymptotically Lifshitz backgrounds, their hyperscaling violating versions and various string theory embeddings we refer the reader to the following recent papers and references therein [11–15]. The literature primarily relevant to us here though concerns earlier work on holographic renormalization and the holographic dictionary for asymptotically Lifshitz backgrounds. In particular, holography for the Einstein–Proca theory with Lifshitz boundary conditions has been discussed from a bottom up perspective in [16–23], while in [24–30] AdS embeddings (or limits) of Lifshitz backgrounds were utilized in order to deduce the non-relativistic dictionary from the relativistic one in special cases.

Our aim in this Letter and the accompanying main paper [10] is to present a general algorithm for the construction of the holographic dictionary of non-relativistic theories, that can be applied to theories with or without a UV fixed point and for any value of the dynamical exponents that is consistent with the null energy condition. The main tool for deriving the holographic dictionary, which includes the Fefferman–Graham expansions, the

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identification of the sources and the dual operators, as well as the boundary counterterms required to render the variational problem well posed, is a general covariant asymptotic solution of the radial Hamilton–Jacobi (HJ) equation. Our main result is a general and efficient algorithm for the recursive solution of the HJ equation, based on the covariant expansion in eigenfunctions of two commuting operators, which are related to the dilatation operator [31] and its generalization for running theories [32].

2. The model

We consider the class of theories defined by the action

$$S_\xi = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^{d+2}x\sqrt{-g}e^{d\xi\phi} \times \left(R-\alpha_\xi(\partial\phi)^2 - Z_\xi F^2 - W_\xi B^2 - V_\xi\right) + \frac{1}{2\kappa^2}\int_{\partial\mathcal{M}} d^{d+1}x\sqrt{-\gamma}2e^{d\xi\phi}K, \tag{3}$$

where α_ξ and ξ are arbitrary parameters and $Z_\xi(\phi)$, $W_\xi(\phi)$ and $V_\xi(\phi)$ are unspecified functions of the real scalar field ϕ. In order to maintain the $U(1)$ gauge symmetry in the presence of a mass for the vector field we have introduced a Stückelberg field ω so that $B_\mu = A_\mu - \partial_\mu\omega$ is gauge invariant. The parameter ξ has been introduced to allow us to interpret our results in any desired Weyl frame. In particular, the ξ dependence of (3) follows from the Weyl transformation $g \to e^{2\xi\phi}g$ of the $\xi = 0$ (Einstein frame) action. Under such a transformation the various parameters and functions of the action transform as $\alpha_\xi = \alpha - d(d+1)\xi^2$, $Z_\xi(\phi) = e^{-2\xi\phi}Z(\phi)$, $W_\xi(\phi) = W(\phi)$, and $V_\xi(\phi) = e^{2\xi\phi}V(\phi)$, where quantities without the subscript ξ refer to the Einstein frame. The advantage of keeping ξ arbitrary in our analysis is that we can impose Lifshitz boundary conditions in a generic ξ frame and cover both Lifshitz and hvLf boundary conditions in the Einstein frame [10]. In the relativistic case, $z = 1$, this trick has been employed in the study of holography for non-conformal branes [33].

3. Lifshitz and hyperscaling violating Lifshitz

The action (3) admits asymptotically locally Lifshitz solutions of the form

$$ds^2 = dr^2 - e^{2zr}dt^2 + e^{2r}d\vec{x}^2, \qquad B = \frac{\mathcal{Q}e^{\epsilon r}}{\epsilon Z_o}dt, \qquad \phi = \mu r, \tag{4}$$

if $V_\xi \sim V_o e^{2(\rho+\xi)\phi}$, $Z_\xi \sim Z_o e^{-2(\xi+\nu)\phi}$, and $W_\xi \sim W_o e^{2\sigma\phi}$ asymptotically, where the parameters of the theory, V_o, Z_o, W_o, ν, ρ, σ, α_ξ, ξ, are related to the Lifshitz boundary condition parameter z and the integration constants μ, ϵ and $\mathcal{Q}$ as

$$\begin{aligned}
&\rho = -\xi, \qquad \nu = -\xi + \frac{\epsilon - z}{\mu}, \qquad \sigma = \frac{z-\epsilon}{\mu},\\
&\mathcal{Q}^2 = \frac{1}{2}Z_o(z-1)\epsilon,\\
&\epsilon = \frac{(\alpha_\xi + d^2\xi^2)\mu^2 - d\mu\xi + z(z-1)}{z-1},\\
&W_o = 2Z_o\epsilon(d+z+d\mu\xi-\epsilon),\\
&V_o = -d(1+\mu\xi)(d+z+d\mu\xi) - (z-1)\epsilon.
\end{aligned} \tag{5}$$

In practice, the parameters we choose to specify at will are $z > 1$, $\alpha > 0$, $Z_o > 0$, ξ and μ. In the Einstein frame (4) are hvLf solutions with $\theta = -d\xi\mu$ and are equivalent to the solutions presented in [13].

4. Radial Hamiltonian formalism

Our starting point for the derivation of the full holographic dictionary is the radial Hamiltonian formalism for the action (3), where the radial coordinate plays the role of Hamiltonian 'time'. Decomposing the bulk fields as

$$\begin{aligned}
&ds^2 = \left(N^2 + N_iN^i\right)dr^2 + 2N_idrdx^i + \gamma_{ij}dx^idx^j,\\
&A = A_rdr + A_idx^i,
\end{aligned} \tag{6}$$

the Hamiltonian takes the form

$$H = \int d^{d+1}x\left(N\mathcal{H} + N_i\mathcal{H}^i + A_r\mathcal{F}\right), \tag{7}$$

where N, N_i and A_r are Lagrange multipliers imposing the Hamiltonian, momentum and $U(1)$ gauge constraints, respectively

$$\begin{aligned}
0 &= \mathcal{H}\\
&= -\frac{\kappa^2}{\sqrt{-\gamma}}e^{-d\xi\phi}\left\{2\left(\pi^{ij}\pi_{ij} - \frac{1}{d}\pi^2\right) + \frac{1}{2\alpha}(\pi_\phi - 2\xi\pi)^2 + \frac{1}{4}Z_\xi^{-1}(\phi)\pi^i\pi_i + \frac{1}{2}W_\xi^{-1}(\phi)\pi_\omega^2\right\}\\
&\quad + \frac{\sqrt{-\gamma}}{2\kappa^2}e^{d\xi\phi}\left(-R[\gamma] + \alpha_\xi\partial^i\phi\partial_i\phi + Z_\xi(\phi)F^{ij}F_{ij} + W_\xi(\phi)B^iB_i + V_\xi(\phi)\right),\\
0 &= \mathcal{H}^i = -2D_j\pi^{ji} + F^i{}_j\pi^j + \pi_\phi\partial^i\phi - B^i\pi_\omega,\\
0 &= \mathcal{F} = -D_i\pi^i + \pi_\omega.
\end{aligned} \tag{8}$$

These constraints provide a full description of the dynamics in the Hamilton–Jacobi (HJ) formalism – there is no need to use the second order equations of motion. This is achieved by expressing the canonical momenta in two different ways. Firstly, they are written as gradients of the Hamilton's principal function $\mathcal{S}[\gamma, A, \phi, \omega]$ as

$$\pi^{ij} = \frac{\delta\mathcal{S}}{\delta\gamma_{ij}}, \qquad \pi^i = \frac{\delta\mathcal{S}}{\delta A_i}, \qquad \pi_\phi = \frac{\delta\mathcal{S}}{\delta\phi}, \qquad \pi_\omega = \frac{\delta\mathcal{S}}{\delta\omega}, \tag{9}$$

and the constraints (8) are interpreted as functional partial differential equations (PDEs) for $\mathcal{S}[\gamma, A, \phi, \omega]$. However, only the Hamiltonian constraint provides a non-trivial dynamical equation. The momentum constraint simply requires that $\mathcal{S}$ be invariant with respect to diffeomorphisms on the radial slice, while the $U(1)$ constraint implies that $\mathcal{S}$ depends on A_i and ω only through the gauge-invariant field B_i. Once a complete integral of these PDEs is known, equating the gradients (9) with the standard expressions for the momenta in terms of the velocities leads to first order flow equations that can be integrated to obtain the full radial dependence of the fields. We refer to [10] for the full set of flow equations for our model.

The radial Hamiltonian formulation of the dynamics is particularly suited for developing the holographic dictionary, both for asymptotically AdS and non-AdS backgrounds. The radial coordinate plays the role of energy scale in the dual field theory, and so it is natural that it is singled out. From the point of view of the bulk theory, the presence of the boundary naturally gives rise to a Gaussian normal radial coordinate. Moreover, the functional $\mathcal{S}$ is precisely the on-shell action, which is interpreted holographically as the generating function of connected correlation functions. The long distance divergences of the on-shell action correspond to a certain asymptotic solution of the HJ equation. The boundary term required to remove these divergences can be defined in terms of such an asymptotic solution of the