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# Some exact solutions of the semilocal Popov equations

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ABSTRACT

We study the semilocal version of Popov's vortex equations on $S^2$. Though they are not integrable, we construct two families of exact solutions which are expressed in terms of rational functions on $S^2$. One family is a trivial embedding of Liouville-type solutions of the Popov equations obtained by Manton, where the vortex number is an even integer. The other family of solutions is constructed through a field redefinition which relates the semilocal Popov equation to the original Popov equation but with the ratio of radii $\sqrt{3/2}$, which is not integrable. These solutions have vortex number $N = 3n - 2$ where $n$ is a positive integer, and hence $N = 1$ solutions belong to this family. In particular, we show that the $N = 1$ solution with reflection symmetry is the well-known $CP^1$ lump configuration with unit size where the scalars lie on $S^3$ with radius $\sqrt{3/2}$. It generates the uniform magnetic field of a Dirac monopole with unit magnetic charge on $S^2$.



Recently, Popov [1] obtained a set of vortex-type equations on a 2-sphere by dimensional reduction of SU(1, 1) Yang–Mills instanton equations on the four-manifold $S^2 \times H^2$ where $H^2$ is a hyperbolic plane. It was shown that they are integrable when the scalar curvature of the manifold vanishes. Subsequently, Manton [2] constructed explicit solutions with even vortex numbers from rational functions on the sphere. They have a geometric interpretation in terms of conformal rescalings of the 2-sphere metric.

The Popov equations involve a complex scalar field and a U(1) gauge potential. Except a flipped sign, they are the same as the well-known Bogomolny equations [3] for abelian Higgs vortices [4,5] on $S^2$. In this paper, we would like to consider the semilocal [6,7] version of the Popov equations, which consist of two scalar fields instead of one. The equations have an additional global SU(2) symmetry with respect to the rotation of the scalars as well as the local U(1) symmetry. We will show that they appear in $2 + 1$ dimensional Chern–Simons systems with nonrelativistic matter on $S^2$. Such systems on the plane have been extensively studied to understand the quantum Hall effect and other related phenomena [8–10]. Then we construct two families of exact solutions of the semilocal Popov equations. One family of solutions is trivially obtained by a simple ansatz that the two scalars are proportional to each other, with which the equations reduce to the original Popov equations. For the other family of solutions, we will relate the equations to the semilocal version of the Liouville equations considered in [11,12]. In addition to Liouville solutions, they admit another family of exact solutions [12] which involves an arbitrary rational function on $S^2$. We will construct solutions of the semilocal Popov equations from them.

It turns out that semilocal Popov equations have another connection to the original Popov equations with a single scalar. As mentioned above, it is integrable only when the scalar curvature of the underlying four-manifold $S^2 \times H^2$ vanishes [1], which happens for equal radii $R_1 = R_2$, where $R_1$, $R_2$ are the radii of $S^2$ and $H^2$, respectively. Here we will show that the semilocal Popov equations with equal radii can be transformed to the Popov equations with different radii $R_1/R_2 = \sqrt{3/2}$. The aforementioned solutions of the semilocal equation correspond to the constant solution of the latter.

The Liouville solutions have only even vortex numbers [2]. However the vortex number of the other family of solutions is $N = 3n - 2$, where $n$ is a positive integer, so that odd vortex numbers are possible. In particular, the solutions with unit vorticity $N = 1$ belong to this family. We will show that the $N = 1$ solution with reflection symmetry in the equator of $S^2$ is precisely given by the $CP^1$ lump configuration with unit size. The $S^3$ where the scalar fields lie has radius $\sqrt{3/2}$ which is the ratio $R_1/R_2$ above. The magnetic field is that of a Dirac monopole with unit magnetic charge on $S^2$.

Let us begin with writing the Popov equations on $S^2$. For convenience, the radius of $S^2$ is fixed to be $\sqrt{2}$. The metric of $S^2$ is given by $ds^2 = \Omega dz d\bar{z}$ with

$$\Omega = \frac{8}{(1 + |z|^2)^2}. \tag{1}$$

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The Popov equations are[1]

$$D_{\bar z}\phi \equiv \partial_{\bar z}\phi - ia_{\bar z}\phi = 0, \tag{2}$$

$$F_{z\bar z} = -\frac{2i}{(1+|z|^2)^2}\left(C^2 - |\phi|^2\right), \tag{3}$$

where $C = R_1/R_2 = \sqrt{2}/R_2$ is the ratio of radii as described above. $\phi$ is a complex scalar field, $a$ is a U(1) gauge potential and $F_{z\bar z} = \partial_z a_{\bar z} - \partial_{\bar z} a_z$ is the field strength which is imaginary. If the right hand side of the second equation has opposite sign, these would be the same as the Bogomolny equations for abelian Higgs vortices on $S^2$. As mentioned above, the equations are integrable only for $C = 1$ [1]. From (2) the gauge potential $a_{\bar z}$ may be expressed as

$$a_{\bar z} = -i\partial_{\bar z}\ln\phi, \tag{4}$$

away from zeros of $\phi$. Since

$$F_{z\bar z} = -i\partial_z\partial_{\bar z}\ln|\phi|^2, \tag{5}$$

we can eliminate the gauge potential and are left with a single equation

$$\partial_z\partial_{\bar z}\ln|\phi|^2 = \frac{2}{(1+|z|^2)^2}\left(C^2 - |\phi|^2\right), \tag{6}$$

which is valid away from zeros of $\phi$.

Eqs. (2) and (3) may be obtained from an energy function [1,2] which comes from a dimensional reduction of the Yang–Mills action,

$$\begin{aligned} E &= \frac{1}{2}\int_{S^2}\left[\frac{4}{\Omega}|F_{z\bar z}|^2 - 2\left(|D_z\phi|^2 + |D_{\bar z}\phi|^2\right)\right. \\ &\quad \left. + \frac{\Omega}{4}\left(C^2 - |\phi|^2\right)^2\right]\frac{i}{2}dz\wedge d\bar z \\ &= \frac{1}{2}\int_{S^2}\left\{-\frac{4}{\Omega}\left[F_{z\bar z} + i\frac{\Omega}{4}\left(C^2 - |\phi|^2\right)\right]^2\right. \\ &\quad \left. - 4|D_{\bar z}\phi|^2\right\}\frac{i}{2}dz\wedge d\bar z - \pi C^2 N, \end{aligned} \tag{7}$$

where $N$ is the first Chern number

$$N = \frac{1}{2\pi}\int_{S^2} F_{z\bar z}\,dz\wedge d\bar z, \tag{8}$$

which is an integer and is the same as the vortex number which counts the number of isolated zeros of $\phi$. Therefore, for fields satisfying the Popov equations, the energy is stationary and has value $-\pi C^2 N$. It is however not minimal because of the negative sign in the second term of (7).

The Popov equation (6) can also arise in a completely different physics system. Let us consider a $2+1$ dimensional Chern–Simons gauge theory with a nonrelativistic matter field on $S^2$ of which the action is

$$\begin{aligned} S &= \int dt\int_{S^2}\left[\frac{\kappa}{2}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \Omega\left(i\phi^* D_t\phi - V\right)\right. \\ &\quad \left. - \left(|\tilde D_z\phi|^2 + |\tilde D_{\bar z}\phi|^2\right)\right]\frac{i}{2}dz\wedge d\bar z, \end{aligned} \tag{9}$$

where $\kappa$ is the Chern–Simons coefficient and

$$\begin{aligned} D_t\phi &= (\partial_t - ia_t)\phi \\ \tilde D_z\phi &= \left(\partial_z - ia_z - iA_z^{ex}\right)\phi. \end{aligned} \tag{10}$$

Note that we applied an external U(1) gauge potential $A^{ex}$ given by

$$A_{\bar z}^{ex} = \frac{i}{2}\frac{gz}{1+|z|^2}, \tag{11}$$

which generates uniform magnetic field with magnetic charge $g$ on $S^2$. The potential $V$ has the form

$$V = -\frac{g}{8}|\phi|^2 + \frac{1}{2\kappa}|\phi|^4. \tag{12}$$

This action has been extensively studied on the plane in the context of anyon physics to understand the quantum Hall effect and other related phenomena [9,10].

Variation of $a_t$ gives the Gauss constraint

$$F_{z\bar z} = -i\frac{\Omega}{2\kappa}|\phi|^2. \tag{13}$$

The energy function is

$$E = \int_{S^2}\left(|\tilde D_z\phi|^2 + |\tilde D_{\bar z}\phi|^2 + \Omega V\right)\frac{i}{2}dz\wedge d\bar z, \tag{14}$$

which has no explicit contribution from the Chern–Simons term. It can be rewritten by the usual Bogomolny rearrangement

$$\begin{aligned} |\tilde D_z\phi|^2 &= |\tilde D_{\bar z}\phi|^2 - i\left(F_{z\bar z} + F_{z\bar z}^{ex}\right)|\phi|^2 \\ &= |\tilde D_{\bar z}\phi|^2 - \frac{\Omega}{2\kappa}|\phi|^4 + \frac{g}{8}\Omega|\phi|^2, \end{aligned} \tag{15}$$

up to a total derivative term, where in the second line we have used (13) and $F_{z\bar z}^{ex} = \frac{ig}{8}\Omega$. The last two terms in (15) are cancelled by the potential (12) and the energy becomes

$$E = 2\int_{S^2}|\tilde D_{\bar z}\phi|^2\frac{i}{2}dz\wedge d\bar z, \tag{16}$$

which is positive definite. Therefore the energy vanishes if

$$\tilde D_{\bar z}\phi = 0. \tag{17}$$

Combining this equation with the Gauss constraint (13), we get

$$\partial_z\partial_{\bar z}\ln|\phi|^2 = -\frac{\Omega}{2\kappa}\left(\frac{\kappa g}{4} - |\phi|^2\right), \tag{18}$$

away from zeros of $\phi$. With $\kappa = -2$ and $g = -2C^2$ this becomes the Popov equation (6).

Now we introduce the semilocal Popov equations which involve two scalar fields $\phi_i$ $(i = 1, 2)$. We will only consider the case of equal radii, i.e., $C = 1$. The semilocal Popov equations read

$$D_{\bar z}\phi_i \equiv \partial_{\bar z}\phi_i - ia_{\bar z}\phi_i = 0, \quad (i = 1, 2) \tag{19}$$

$$F_{z\bar z} = -\frac{2i}{(1+|z|^2)^2}\left(1 - |\phi_1|^2 - |\phi_2|^2\right), \tag{20}$$

which have an obvious global SU(2) symmetry. These equations can again be obtained from the energy function generalizing (7) by introducing two scalars

$$\begin{aligned} E &= \frac{1}{2}\int_{S^2}\left\{\frac{(1+|z|^2)^2}{2}|F_{z\bar z}|^2 - 2\sum_{i=1}^{2}\left(|D_z\phi_i|^2 + |D_{\bar z}\phi_i|^2\right)\right. \\ &\quad \left. + \frac{2}{(1+|z|^2)^2}\left(1 - |\phi_1|^2 - |\phi_2|^2\right)^2\right\}\frac{i}{2}dz\wedge d\bar z. \end{aligned} \tag{21}$$

[1] We follow the notation of [2].