Available online at www.sciencedirect.com
ScienceDirect
Physics Letters B 658 (2008) 222–229
PHYSICS LETTERS B
www.elsevier.com/locate/physletb

Third-order non-Coulomb correction to the S-wave quarkonium wave functions at the origin

M. Beneke [a,*], Y. Kiyo [b], K. Schuller [a]

[a] *Institut für Theoretische Physik E, RWTH Aachen, D-52056 Aachen, Germany*
[b] *Institut für Theoretische Teilchenphysik, Universität Karlsruhe, D-76128 Karlsruhe, Germany*

Received 20 June 2007; received in revised form 10 September 2007; accepted 11 September 2007

Available online 17 November 2007

Editor: A. Ringwald

Abstract

We compute the third-order correction to the S-wave quarkonium wave functions $|\psi_n(0)|^2$ at the origin from non-Coulomb potentials in the effective non-relativistic Lagrangian. Together with previous results on the Coulomb correction and the ultrasoft correction computed in a companion paper, this completes the third-order calculation up to a few unknown matching coefficients. Numerical estimates of the new correction for bottomonium and toponium are given.

1. Introduction

The non-relativistic bound-state problem has a long history since the birth of quantum mechanics. Its systematic derivation from the relativistic quantum field theory of electrodynamics or chromodynamics (QCD) was developed more recently. Non-relativistic effective field theories [1–3] together with dimensional regularization and diagrammatic expansion methods [4] now allow calculations of higher-order perturbative corrections, including all "relativistic" effects, to the leading-order bound-state properties, given by the solution of the Schrödinger equation. This is of interest in QCD for the lowest bottomonium state and top–antitop production near threshold, where non-perturbative long-distance effects can be argued to be sub-dominant, but perturbative corrections are large.

The S-wave energy levels are currently known at next-to-next-to-next-to-leading order (NNNLO)[1] [5–8], except for the three-loop coefficient of the colour-Coulomb potential, but the corresponding wave functions at the origin, which are related to electromagnetic decay and production of these states are completely known only at next-to-next-to-leading order (NNLO) [9–11]. There exist partial results for logarithmic effects at NNNLO [12–15], which can be related to certain anomalous dimensions and lower-order quantities. In [7] we computed the third-order corrections to S-wave wave function at the origin from all terms in the heavy-quark potential related only to the Coulomb potential. In this Letter we compute the contribution from the remaining potentials. A companion paper [16] deals with the Lamb-shift like contribution from ultrasoft gluons, thus completing the calculation of all bound-state effects at NNNLO, except for a few unknown matching coefficients. Our result is provided in such a form that these coefficients can be easily inserted, once they are computed.

In contrast to the Coulomb corrections the calculation of the more singular non-Coulomb potential corrections leads to divergences, both in the calculation of the potentials themselves as in the insertions of these potentials in the calculation of the wave

* Corresponding author.
E-mail address: mbeneke@physik.rwth-aachen.de (M. Beneke).

[1] Non-relativistic perturbation theory is an expansion in α_s and the non-relativistic velocity v, while counting $\alpha_s/v \sim 1$, which implies a summation of the series in α_s even at LO. We do not sum logarithms of $\alpha_s \ln v$.

0370-2693/$ – see front matter
doi:10.1016/j.physletb.2007.09.074

function at the origin. We employ dimensional regularization with $d = 4 - 2\epsilon$ throughout, and provide a precise definition of all quantities, which corresponds to the $\overline{\text{MS}}$ subtraction scheme. The technical details of this calculation together with an extension to the full S-wave Green function will be given elsewhere.

2. Relating the leptonic quarkonium decay constant to the wave function at the origin

We consider the two-point function

$$\left(q^\mu q^\nu - g^{\mu\nu} q^2\right)\Pi\left(q^2\right) = i \int d^d x\, e^{iqx} \langle \Omega | T\left(j^\mu(x) j^\nu(0)\right)|\Omega\rangle, \tag{1}$$

of the electromagnetic heavy-quark current $j^\mu = \bar{Q}\gamma^\mu Q$, choosing $q^\mu = (2m + E, \mathbf{0})$ with m the pole mass of the heavy quark. The two-point function exhibits the S-wave bound-state poles at E_n, near which

$$\Pi\left(q^2\right) \overset{E\to E_n}{=} \frac{N_c}{2m^2} \frac{Z_n}{E_n - E - i\epsilon}. \tag{2}$$

Here $N_c = 3$ denotes the number of colours. The residue Z_n is related to the leptonic decay width $\Gamma([Q\bar{Q}]_n \to l^+ l^-)$ of the nth S-wave quarkonium state by

$$\Gamma\left([Q\bar{Q}]_n \to l^+ l^-\right) = \frac{4\pi N_c e_Q^2 \alpha^2 Z_n}{3m^2}, \tag{3}$$

with e_Q the electric charge of the heavy quark in units of the positron charge, and α the fine-structure constant. Although there are no toponium states, and the cross section of top–antitop production is determined by the full two-point function, the residue Z_n for $n = 1$ provides an approximation to the height of the broad resonance in this cross section.

The electromagnetic current j^μ is expressed in terms of the non-relativistic heavy quark (ψ) and antiquark (χ) field operators via

$$j^i = c_v \psi^\dagger \sigma^i \chi + \frac{d_v}{6m^2} \psi^\dagger \sigma^i \mathbf{D}^2 \chi + \cdots, \tag{4}$$

where the hard matching coefficients have expansions $c_v = 1 + \sum_n c_v^{(n)} (\alpha_s/4\pi)^n$, and the $d_v = 1 + d_v^{(1)}(\alpha_s/4\pi) + \cdots$. The central quantity in this Letter is the two-point function

$$G(E) = \frac{i}{2N_c(d-1)} \int d^d x\, e^{iEx^0} \langle \Omega | T\left(\left[\psi^\dagger \sigma^i \chi\right](x)\left[\chi^\dagger \sigma^i \psi\right](0)\right)|\Omega\rangle \overset{E\to E_n}{=} \frac{|\psi_n(0)|^2}{E_n - E - i\epsilon}, \tag{5}$$

defined in non-relativistic QCD (NRQCD), whose poles define the wave functions at the origin and bound-state energy levels. At leading order, the wave functions and binding energies are given by $|\psi_n^{(0)}(0)|^2 = (mC_F\alpha_s)^3/(8\pi n^3)$ and $E_n^{(0)} = -m(\alpha_s C_F)^2/(4n^2)$, respectively (here and below $C_F = (N_c^2 - 1)/(2N_c) = 4/3$, $C_A = N_c = 3$). They receive perturbative corrections from higher-order heavy-quark potentials and dynamical gluon effects, hence $E_n = E_n^{(0)}(1 + \sum_k (\alpha_s/4\pi)^k e_k)$ and $|\psi_n(0)|^2 = |\psi_n^{(0)}(0)|^2(1 + \sum_k (\alpha_s/4\pi)^k f_k)$. Using an equation-of-motion relation, we can replace $\mathbf{D}^2$ in (4) by $-mE$, and we obtain

$$Z_n = c_v\left[c_v - \frac{E_n}{m}\left(1 + \frac{d_v}{3}\right) + \cdots\right]\left|\psi_n(0)\right|^2, \tag{6}$$

where terms beyond NNNLO are neglected. Inserting the perturbative expansions and defining $Z_n = |\psi_n^{(0)}(0)|^2(1 + \sum_k (\alpha_s/4\pi)^k z_k)$, results in

$$z_1 = 2c_v^{(1)} + f_1, \tag{7}$$

$$z_2 = 2c_v^{(2)} + c_v^{(1)2} + 2c_v^{(1)} f_1 + f_2 - \frac{4}{3}\frac{16\pi^2 E_n^{(0)}}{m\alpha_s^2}, \tag{8}$$

$$z_3 = 2c_v^{(3)} + 2c_v^{(1)}\left(c_v^{(2)} + f_2\right) + \left(2c_v^{(2)} + c_v^{(1)2}\right) f_1 + f_3 - \frac{16\pi^2 E_n^{(0)}}{m\alpha_s^2}\left[\frac{d_v^{(1)}}{3} + \frac{4}{3}\left(c_v^{(1)} + e_1 + f_1\right)\right]. \tag{9}$$

Note that e_k, f_k and z_k depend on the principal quantum number n of the energy level, but we omitted a corresponding index to keep the notation short. The short-distance coefficients $c_v^{(1)}$, $c_v^{(2)}$ in the $\overline{\text{MS}}$ scheme[2] are given in [17,18]. The third-order coefficient $c_v^{(3)}$

[2] The $\overline{\text{MS}}$ scheme is defined by the loop integration measure $\tilde{\mu}^{2\epsilon} d^d k/(2\pi)^d$ with $\tilde{\mu}^2 = \mu^2 e^{\gamma_E}/(4\pi)$ and subtraction of the pole parts in ϵ.