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Differential structure on the κ-Minkowski spacetime from twist

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ABSTRACT

We study four-dimensional κ-Minkowski spacetime constructed by the twist deformation of $U(igl(4,R))$. We demonstrate that the differential structure of such twist-deformed κ-Minkowski spacetime is closed in four dimensions contrary to the construction of κ-Poincaré bicovariant calculus which needs an extra fifth dimension. Our construction holds in arbitrary dimensional spacetimes.

1. Introduction

There has been much interest in recent years in a possible role of deformation of spacetime symmetry in describing Planck scale physics. In particular, initiated by the κ-deformed Poincaré algebra [1] the κ-Minkowski spacetime [2,3] satisfying

$$[x^0, x^i] = \frac{i}{\kappa} x^i, \qquad [x^i, x^j] = 0, \tag{1}$$

has attracted much attention in explaining cosmic observational data, since the deformation preserves the rotational symmetry in space. The differential structure of the κ-Minkowski spacetime has been constructed in [4] and based on this differential structure, the scalar field theory has been formulated [5–8]. Similar field theoretic approach is given in [9,10] using the coproduct and star product as Lie-algebraic noncommutative spacetime. It was shown that the differential structure requires that the momentum space corresponding to the κ-Minkowski spacetime becomes a de Sitter section in five-dimensional flat space. The κ-deformation was extended to the curved space with κ-Robertson–Walker metric and was applied to the cosmic microwave background radiation in [11]. The physical effects of the κ-deformation has been investigated on the Unruh effect [7], black-body radiation [12] and Casimir effect [13]. The Fock space and its symmetries [14,15], κ-deformed statistics of particles [16,17], and interpretation of the κ-Minkowski spacetime in terms of exotic oscillator [18] were also studied.

Recently, simpler realization of the κ-Minkowski spacetime by the use of twisting procedure have been sought by several authors [19–23]. It happens that only the case of the light-cone κ-deformation the deformed Poincaré algebra can be described by standard twist (see e.g. [19]).

By embedding an Abelian twist in $IGL(4,R)$ whose symmetry is larger than the Poincaré, the realization for the time-like κ-deformation was first constructed in Ref. [20] and then by [21]. Some physical properties of analogous twist realization of κ-Minkowski spacetime were discussed recently [22]. This approach can be seen as an alternative to the κ-like deformation of the quantum Weyl and conformal algebra [23], which is obtained by using the Jordanian twist [24]. One may even consider the chains of twists for classical Lie algebras [25].

In this Letter, we will construct the κ-Minkowski spacetime and its differential structure using the twisted universal enveloping Hopf algebra of the inhomogeneous general linear group in $(3+1)$-dimensions. In Section 2, the κ-Minkowski spacetime from twist is reviewed and in Section 3, its differential structure is constructed. We show that the differential structure is closed without extra dimension.

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2. Review on the κ-Minkowski spacetime from twist

Twisting the Hopf algebra of the universal enveloping algebra of $igl(4, R)$ is considered in [20,21]. The group of inhomogeneous linear coordinate transformations is composed of the product of the general linear transformations and the spacetime translations. The inhomogeneous general linear algebra in $(3+1)$-dimensional flat spacetime $\mathbf{g} = igl(4, R)$ is composed of 20 generators $\{P_a, M^a{}_b\}$ $(a, b = 0, 1, 2, 3)$ where P_a represents the spacetime translation and $M^a{}_b$ homogeneous one including the boost generator, rotation and dilation. The generators satisfy the commutation relations,

$$[P_a, P_b] = 0, \qquad [M^a{}_b, P_c] = i\delta^a{}_c \cdot P_b,$$
$$[M^a{}_b, M^c{}_d] = i(\delta^a{}_d \cdot M^c{}_b - \delta^c{}_b \cdot M^a{}_d). \tag{2}$$

The universal enveloping Hopf algebra $\mathcal{U}(\mathbf{g})$ can be constructed starting from the base elements $\{1, P_a, M^a{}_b\}$ and coproduct $\Delta Y = 1 \otimes Y + Y \otimes 1$ with $Y \in \{P_a, M^a{}_b\}$. The operators representing energy E and spatial dilatation D is defined by

$$E = P_0, \qquad D = \sum_{i=1}^{3} M^i{}_i. \tag{3}$$

Note that the two generators D and E commutes with each other, $[D, E] = 0$. In Ref. [20], an Abelian twist element $\mathcal{F}_\kappa$,

$$\mathcal{F}_\kappa = \exp\left[\frac{i}{\kappa}\big(\alpha E \otimes D - (1-\alpha) D \otimes E\big)\right], \tag{4}$$

is shown to generate the κ-Minkowski spacetime with the twisted Hopf algebra $\mathcal{U}_\kappa(\mathbf{g})$. α is a constant chosen as $\alpha = 1/2$ in this Letter which corresponds to the symmetric ordering of the exponential kernel function in the conventional κ-Minkowski spacetime formulation. Other choice of α represents a different ordering.

Co-unit and antipode are not twisted $\epsilon_{\mathcal{F}} = \epsilon$ and $S_{\mathcal{F}} = S$, but coproduct is twisted as

$$\Delta_\kappa(Y) = \mathcal{F}_\kappa \cdot \Delta Y \cdot \mathcal{F}_\kappa^{-1} = \sum_i Y_{(1)i} \otimes Y_{(2)i} \equiv Y_{(1)} \otimes Y_{(2)}. \tag{5}$$

Explicitly $(i, j = 1, 2, 3)$,

$$\Delta_\kappa(Z) = Z \otimes 1 + 1 \otimes Z, \quad Z \in \{E, D, M^i{}_j\},$$
$$\Delta_\kappa(P_i) = P_i \otimes e^{E/(2\kappa)} + e^{-E/(2\kappa)} \otimes P_i,$$
$$\Delta_\kappa(M^i{}_0) = M^i{}_0 \otimes e^{-E/(2\kappa)} + e^{E/(2\kappa)} \otimes M^i{}_0,$$
$$\Delta_\kappa(M^0{}_i) = M^0{}_i \otimes e^{E/(2\kappa)} + e^{-E/(2\kappa)} \otimes M^0{}_i + \frac{1}{2\kappa}\big(P_i \otimes D e^{E/(2\kappa)} - e^{-E/(2\kappa)} D \otimes P_i\big),$$
$$\Delta_\kappa(M^0{}_0) = M^0{}_0 \otimes 1 + 1 \otimes M^0{}_0 + \frac{1}{2\kappa}(E \otimes D - D \otimes E). \tag{6}$$

It is noted that the twisted Hopf algebra is different from that of the conventional κ-Poincaré algebra in two aspects. First, the algebraic part is nothing but those of the un-deformed inhomogeneous general linear group (2) rather than that of the deformed Poincaré. Second, the co-algebra structure is enlarged due to the bigger symmetry $igl(4)$ and its co-product is deformed as (6).

3. Differential structure

The inhomogeneous general linear group $IGL(4, R)$ acts on the coordinate space $\{x^a\}$ and the twisted-coproduct of the generator Y acts on the tensor product space of $\{x^a \otimes x^b\}$. Thus, one can define the $*$-product of the coordinate vectors x^a in terms of the twist action on the coordinates. Explicitly,

$$x^a * x^b \equiv *[x^a \otimes x^b] = \cdot\big[\mathcal{F}_\kappa^{-1} \triangleright (x^a \otimes x^b)\big]. \tag{7}$$

This results in the noncommutative commutation relation of the coordinates

$$[x^0, x^j]_\kappa \equiv x^0 * x^j - x^j * x^0 = \frac{i}{\kappa} x^j, \qquad [x_i, x_j]_\kappa = 0,$$

which reproduces the commutation relation (1).

To understand the differential structure, one has to incorporate the (co-)tangent space and investigate the action of $IGL(4, R)$ on the space. Suppose that one constructs a set of basis vectors of a coordinate system $CS = \{e_a \mid a = 0, 1, 2, 3\}$ of the four-dimensional vector space V_4 which are not necessarily ortho-normal. One naturally demands that the homogeneous transformation Λ acts on the coordinates x^a, the dual-basis of the coordinate system e^a, and a function f as

$$\Lambda: \quad \begin{cases} x^a \to x^{a'};\ x^{a'} = x^b \Lambda_b^{a'}, \\ e^a \to e^{a'};\ e^{a'} = e^b \Lambda_b^{a'}, \\ f \to f';\ f'(x') = f(x) = f\big(x^{b'} (\Lambda^{-1})^a_{b'}\big), \end{cases} \tag{8}$$

and the translation T by the amount of coordinate vector y^a as

$$T(y^a): \quad \begin{cases} x^a \to x^{a'};\ x^{a'} = x^a + y^a, \\ e^a \to e^{a'};\ e^{a'} = e^a, \\ f \to f';\ f'(x^{a'}) = f(x^a) = f(x^{a'} - y^a). \end{cases} \tag{9}$$

Then, the infinitesimal transformation is given in terms of $igl(4, R)$ generators:

$$\delta_\epsilon S = -i\epsilon^c Y_c \triangleright S. \tag{10}$$

The action of $M^a{}_b$ is represented by

$$M^a{}_b \triangleright x^c = -ix^a \delta^c_b, \qquad M^a{}_b \triangleright e^c = -ie^a \delta^c_b,$$
$$\big(M^a{}_b \triangleright f\big)(x^c) = -ix^a \frac{\partial}{\partial x^b} f(x^c), \tag{11}$$

and of P_a by

$$P_a \triangleright x^b = -i\delta^b_a, \qquad P_a \triangleright e^b = 0,$$
$$(P_a \triangleright f)(x^b) = -i\frac{\partial}{\partial x^a} f(x^b). \tag{12}$$

Note that the translation and thus, the energy operator E does not change the dual basis vector e^a. On the other hand, the spatial dilatation operator D non-trivially acts as:

$$\exp(i\alpha D) \triangleright x^a = x^a_{(\alpha)} \exp(i\alpha D)\triangleright,$$
$$\exp(i\alpha D) \triangleright e^a = e^a_{(\alpha)} \exp(i\alpha D)\triangleright,$$
$$\big(\exp(i\alpha D) \triangleright f\big)(x^a) = f\big(x^a_{(\alpha)}\big),$$

where $x^a_{(\alpha)} = (x^0, \exp(\alpha) x^i)$ and $e^a_{(\alpha)} = (e^0, \exp(\alpha) e^i)$. This nontrivial transformation law provides the $*$-product between the space coordinates and/or the dual-basis vectors $\{x^a, e^a\}$. Between the two basis vectors, we have

$$e^a * e^b = \cdot\big[\mathcal{F}_\kappa^{-1}(e^a \otimes e^b)\big] = e^a e^b,$$

where the time translational invariance (12) $E \triangleright e^a = 0$ is used. Between e^a and x^b we have

$$e^a * x^b = m\big[\mathcal{F}_\kappa^{-1}(e^a \otimes x^b)\big] = e^a x^b - \frac{i}{2\kappa}\delta^a_i \delta^b_0 e^i,$$
$$x^b * e^a = m\big[\mathcal{F}_\kappa^{-1}(x^b \otimes e^a)\big] = e^a x^b + \frac{i}{2\kappa}\delta^a_i \delta^b_0 e^i,$$

which results in the commutation relation

$$[e^a, e^b]_\kappa = 0, \qquad [x^0, e^i]_\kappa = \frac{i}{\kappa} e^i,$$
$$[x^0, e^0]_\kappa = 0 = [x^i, e^a]_\kappa. \tag{13}$$