Analytic perturbation theory versus $1/N$ expansion in the Gross–Neveu model

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Abstract

The $1/N$ expansion can be successfully used to calculate the Green functions of the two-dimensional $O(2N)$ Gross–Neveu model. In parallel, the methods of analytic perturbation theory are also applied. Comparing the results of these two calculations at leading order, we report on the surprising agreement between them.
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An analytic model for the QCD running coupling $\bar{\alpha}_s(Q^2)$ has been proposed in [1]. Since then, this concept evolved to become the method known as analytic perturbation theory (APT) [2]. The model is essentially based on the old idea of combining the renormalization group (RG) summation of the perturbative series of ultra-violet (UV) logarithms with analyticity in the Q^2 variable. This idea has been used previously in the context of the QED ghost-pole problem [3]. Certain reservations are in order when applying this idea to the QCD case. Some of them have been pointed out in [1]. However, this discussion did not address the pertinent aspect of QCD which relates to chiral symmetry breaking (ChSB) in the infra-red (IR) region [4]. It is commonly accepted that this nonperturbative phenomenon takes place in QCD.

Recently attempts have been undertaken to apply a second order Bethe–Salpeter formalism to the evaluation of the meson spectrum in QCD [5]. The idea of broken chiral symmetry was taken into account through an approximate solution of the appropriate Dyson–Schwinger equations. It has been shown that the meson spectrum is well described if the analytic running

coupling $\bar{\alpha}_{\mathrm{APT}}(Q^2)$ is used in the region $Q^2 < 1\ \mathrm{GeV}^2$, i.e., it was assumed there that except for analyticity, the IR behaviour of this coupling does not require any additional nonperturbative corrections, caused for instance by the dynamical ChSB effect in QCD.

The analyticity of the running coupling function is a general requirement applied to any field theory, including the theories with dynamical ChSB. The latter however can possess an interesting property: there are models [6] where the analyticity of the running coupling is automatically fulfilled in the true ground state. On the other hand, the same coupling, being calculated in the false vacuum, possesses a Landau singularity in the IR region of Q^2. At large momenta both calculations coincide.

Let us imagine for a moment that one knows only the result of the false vacuum calculation. It is natural in this case to use the APT method. The corresponding analyticization procedure, based on the spectral representation of the Källen–Lehmann type, gives rise to a nonperturbative contribution, which "cures" the IR ghost-pole problem. This improvement in the low-energy part of the theory should nevertheless be consistent with the idea of spontaneously broken symmetry: improved calculations based on the false vacuum should not deviate much from the result obtained from the true ground state. If deviations turn out to be big, one should assume that additional nonperturbative

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contributions must still be considered. On the contrary, if deviations are small the APT result can be a good approximation even for theories with dynamical ChSB.

Certainly, it would be most tempting to use the APT result without introducing additional corrections due to ChSB, i.e., supposing that the nonperturbative correction inherent in analyticity already emulates a major part of the ChSB effect, at least up to some low-energy scale. The attempt to extract the IR QCD coupling from the meson spectrum and to compare it with the APT result has been made recently in [7].

Here we show that there is a sensible way to discuss this point also on pure theoretical grounds by applying the APT method to a theory which possesses crucial properties of QCD and yet is solvable both in the UV and IR regions. To put it more precisely, following the soluble two-dimensional $O(2N)$-symmetric Gross–Neveu (GN) model [8], which is known to be a renormalizable and asymptotically free quantum field theory, we argue in this Letter that there is actually a remarkable agreement between the running coupling obtained by the APT method and, independently, by the $1/N$ nonperturbative expansion.

One should elucidate two points here. First, we compare only the leading order results, believing that higher orders can be interesting if the first step happens to be successful. Second, to be precise one can speak only about accurate agreement in the region of Euclidean momenta $Q^2 > \Lambda^2$, i.e., above the perturbative Landau singularity. It would be too naive to expect that the simple analyticization procedure can perfectly describe all details of the nonperturbative dynamics. Nevertheless, our calculations clearly show that this procedure is compatible with the fact that fermions acquire their masses dynamically. The same conclusion has been made in [7] for QCD.

The effective coupling $\bar{g}^2(Q^2)$ of the four-fermion interactions in the GN model, obtained after an RG resummation of the leading UV logs in the naive vacuum, i.e., for massless fermions, is described by the formula

$$N\bar{g}_{\mathrm{RG}}^{2(1)}\big(Q^2\big) = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)}. \tag{1}$$

The l.h.s. is fixed as $N \to \infty$, this explains the presence of the factor N. Here $\beta_0 = 1/2\pi$, Λ is an RG invariant scale, $\Lambda = \mu \exp[-1/(2\beta_0 g_\mu^2)]$, and g_μ is the coupling constant renormalized at a scale μ (a momentum cut-off scheme is used; dimensional regularization gives essentially the same result).

After imposing APT analyticity one obtains an expression similar to that found for the QCD coupling $\bar{\alpha}_{\mathrm{APT}}(Q^2)$ (see Ref. [1]) at the one loop order in the region $Q^2 > 0$

$$N\bar{g}_{\mathrm{APT}}^{2(1)}\big(Q^2\big) = \frac{1}{\beta_0}\left(\frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2}\right). \tag{2}$$

The last term eliminates the unphysical Landau singularity at $Q^2 = \Lambda^2$ from the perturbative RG improved result (1). This singularity results from the assumption that fermions are massless in the vacuum. We do this on purpose. Our aim at this stage is to obtain the invariant coupling $\bar{g}^2(Q^2)$ by the APT method without detailed description of the phase state of the GN model. In some respect this resembles the QCD case where one obtains

the analogues of Eqs. (1) and (2) neglecting details related to the ChSB effect. As a possible alternative we mention here a model for the QCD invariant coupling, which is based on the hypothesis of finite gluon and quark masses [9].

At present the main properties of the GN model are well studied. In particular, it is known that the $1/N$ expansion is a rather firm method to find that fermions are massive in the true vacuum [8]. Indeed, as has been stressed first by Witten [10], the GN model possesses a low-temperature phase of the Berezinski–Kosterlitz–Thouless type [11]. In this phase, the fermions have mass $M \neq 0$, just as if there were true spontaneous chiral symmetry breaking. It is fortunate (although perhaps just a coincidence) that the model exhibits in this way a feature that would be present in higher dimensions. Here we would like to stress only two important facts in favour of the $1/N$ expansion: (a) it yields the correct spectrum for the model [12]; (b) the exact S-matrix obtained in [13] can be expanded in powers of $1/N$ correctly reproducing the known terms of the $1/N$ series (see also Ref. [14]). It shows that the $1/N$ expansion is a well-convergent series.

Thus, the $1/N$ expansion provides a solid basis for the calculation of the fermion mass M, which results in $M = \Lambda$ at leading order (both the exact and $1/N$ results for the ratio M/Λ can be found in Ref. [15]). To obtain the RG invariant running coupling $\bar{g}^2(Q^2)$ one can use the mass dependent RG method [16]. In the true ground state ($M \neq 0$) one has [6,8]

$$N\bar{g}_{1/N}^{2(1)}\big(Q^2\big) = \frac{1}{\beta_0}\left[X \ln \frac{X+1}{X-1}\right]^{-1}, \tag{3}$$

where $X = \sqrt{1 + 4M^2/Q^2}$. Both expressions, namely, Eqs. (2) and (3), have the same UV asymptotics.

Note also that the fermion mass can be simply found by considering the one-loop effective potential $V(\phi_c)$ of the GN model as a function of a classical scalar field ϕ_c which represents the fermion–antifermion bound state

$$V(\phi_c) = \frac{N\phi_c^2}{4\pi}\left(\ln\frac{\phi_c^2}{\Lambda^2} - 1\right). \tag{4}$$

This function has two stationary points, $\phi_c = 0$ and $\phi_c = \Lambda$, and a positive second derivative, $V''(\Lambda) = N/\pi$ (minimum), for the second case alone. These two solutions correspond to the massless (false vacuum) and massive (true vacuum) fermion states.

The functions given by Eqs. (1), (2), and (3) are presented in Fig. 1. Our first observation is that the two functions $N\bar{g}_{\mathrm{APT}}^{2(1)}(Q^2)$ and $N\bar{g}_{1/N}^{2(1)}(Q^2)$ are quite close to each other in the interval $1 < Q^2/\Lambda^2 < \infty$. In this domain the analytic correction agrees perfectly well with the well-established fact that the fermions have a nonzero mass in the GN model. There is a sizeable difference in the results only at small values of Q^2, i.e., in the IR region $0 < Q^2 < \Lambda^2$. Here, for instance, the value $N\bar{g}_{1/N}^{2(1)}(0) = (2\beta_0)^{-1}$ is smaller than $N\bar{g}_{\mathrm{APT}}^{2(1)}(0)$ by a factor of two, and $\bar{g}_{1/N}^{2(1)}(0) = \bar{g}_{\mathrm{APT}}^{2(1)}(\Lambda^2)$. In this region, the results of the analytic model should be taken with care. This is our second observation.