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Black hole quantum tunnelling and black hole entropy correction

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ABSTRACT

The Parikh–Wilczek tunnelling framework, which treats Hawking radiation as a tunnelling process, is investigated once more in this work. The first order correction, the log-corrected entropy-area relation, emerges naturally in the tunnelling picture if we consider the emission of a spherical shell. The second order correction to the emission rate for the Schwarzschild black hole is also calculated. At this level, the entropy of the black hole will contain three parts: the usual Bekenstein–Hawking entropy, a logarithmic term and an inverse area term. We find that the coefficient of the logarithmic term is -1. Thus, apart from a coefficient, our correction to the black hole entropy is consistent with that calculated in loop quantum gravity.

1. Introduction

In 2000, Parikh and Wilczek proposed an approach for calculating the emission rate at which particles tunnel across the event horizon [1]. They treated Hawking radiation as a tunnelling process, and used the WKB method [2,3]. In this way they calculated a corrected spectrum, which is accurate as a first order approximation. Following this method, many static or stationary rotating black holes have been studied [4–31]. In all of this work, the entropy of the black hole contains only the Bekenstein–Hawking entropy. One may ask: will the Parikh–Wilczek framework still be true if the quantum corrections to the entropy are taken into account? At present, consideration of such quantum corrections has produced model and method dependent results [32–43]. The general expression for the black hole entropy is [44,45]

$$S_q = \frac{A_H}{4l_p^2} + \alpha \ln \frac{A_H}{4l_p^2} + O\left(\frac{l_p^2}{A_H}\right) + \text{const}, \tag{1}$$

where α is a model-dependent (dimensionless) parameter. In the case of Loop Quantum Gravity α is a negative coefficient whose exact value was once an object of debate (see e.g. [37]) but has since been rigorously fixed at $\alpha = -1/2$. In String Theory the sign of α depends on the number of field species appearing in the low energy approximation [36]. It would, therefore, be very interesting work to introduce the log-corrected entropy–area relation in the tunnelling framework. Moreover, one may ask: if the emission rate is calculated to second order, will the entropy contain the inverse area term as given in Eq. (1)? In this Letter we first show that, in the tunnelling picture and taking the emission of a particle in the form of a surface wave (spherical shell), a logarithmic correction term does occur in the expression of the black hole entropy. We then verify that, if we calculate the emission rate to second order using the Parikh–Wilczek tunnelling framework, the entropy of the black hole will contain three parts: the usual Bekenstein–Hawking entropy, the logarithmic term and the inverse area term. We finally make two comments as to the validity of the Parikh–Wilczek framework in our calculation.

2. Black hole tunnelling and the first order correction to the black hole entropy

As mentioned above, Parikh and Wilczek applied the WKB approximation to calculate the emission rate of a tunnelling particle (an S-shell wave). We start with a brief review of the WKB method and barrier penetration. For a massless particle (massless shell), the infinite blueshift near the black hole horizon causes the characteristic wavelength of any wavepacket of the S-wave (see [1–3]) to be arbitrarily small near the horizon. Given this, the geometrical optics limit becomes an especially reliable approximation. The geometrical optics limit allows us to obtain rigorous results in the language of particles directly. That is, the WKB method and the expression of the emission rate are the same as that of a classical massive particle. With this in mind, we only study the tunneling process for a massive particle (massive shell) in what follows.

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Schrödinger's equation for the motion of a particle in a centrally symmetric field is

$$\Delta\psi + (2m/\hbar^2)(E - U(r))\psi = 0. \tag{2}$$

Let us consider the following radial equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{l(l+1)}{r^2}R + \frac{2m}{\hbar^2}(E - U(r))R = 0. \tag{3}$$

By the substitution

$$R(r) = X(r)/r \tag{4}$$

Eq. (3) is brought to the form

$$\frac{d^2X}{dr^2} + \left[\frac{2m}{\hbar^2}(E - U(r)) - \frac{l(l+1)}{r^2}\right]X = 0. \tag{5}$$

For the S-wave, $l = 0$, the equation for $X(r)$ is:

$$\frac{d^2X}{dr^2} + \frac{2m}{\hbar^2}(E - U(r))X = 0. \tag{6}$$

Note that, in the Parikh–Wilczek framework, the tunnelling particle is considered as a spherical shell (S-wave) in order to calculate the particle's self-gravitation reliably. In this way, upon emission from the black hole, the matter–gravity system transitions from one spherical state to another. So, the de Broglie wave function of the emission spherical shell should be:

$$\psi(r) = X(r)/r. \tag{7}$$

That is, the WKB wave function of the particle can be written as

$$\psi(r) = \frac{X(r)}{r} = \frac{1}{r}\exp\left[\frac{iS(r)}{\hbar}\right], \tag{8}$$

where

$$S(r) = S_0(r) + \left(\frac{\hbar}{i}\right)S_1(r) + \left(\frac{\hbar}{i}\right)^2 S_2(r) + \cdots. \tag{9}$$

Substituting (8) into Schrödinger equation (6) yields

$$S_0 = \pm\int^r p_r\,dr, \tag{10}$$

$$2S_0'S_1' + S_0'' = 0, \tag{11}$$

$$2S_0'S_2' + (S_1')^2 + S_1'' = 0, \tag{12}$$

where we use a prime to denote differentiation with respect to r.

To evaluate the probability of a particle passing through the barrier, we divide the whole region of motion of the particle by two tunnelling points, a and b, into three parts: the ingoing and reflecting region I, the barrier region II and the outgoing region III. The particle moves as a free particle in region I and III, but region II is classically inaccessible.

In region I, we take the WKB wave function as follows [46]:

$$X_{\rm I}(r) = \frac{2}{\sqrt{v}}\sin\left[\frac{1}{\hbar}\int_r^a p_r\,dr + \frac{\pi}{4}\right]$$
$$= \frac{1}{i\sqrt{v}}\left\{\exp\left[\frac{i}{\hbar}\int_r^a p_r\,dr + \frac{i\pi}{4}\right] - \exp\left[-\frac{i}{\hbar}\int_r^a p_r\,dr - \frac{i\pi}{4}\right]\right\}, \tag{13}$$

where v is the velocity of the tunnelling particle. In region II, the WKB wave function is a linear combination of real exponentials. Considering the connection between the oscillating and exponential solutions at $r = a$, the WKB wave function in region II can be written as

$$X_{\rm II}(r) = \frac{1}{\sqrt{v}}\exp\left[-\frac{1}{\hbar}\left|\int_a^b p_r\,dr\right|\right]\exp\left[-\frac{1}{\hbar}\left|\int_b^r p_r\,dr\right|\right]. \tag{14}$$

The WKB wave function in region III is:

$$X_{\rm III}(r) = -\frac{1}{\sqrt{v}}\exp\left[-\frac{1}{\hbar}\left|\int_a^b p_r\,dr\right|\right]\exp\left[\frac{i}{\hbar}\int_b^r p_r\,dr + \frac{i\pi}{4}\right]. \tag{15}$$

The probability of barrier penetration is

$$\Gamma_p = \frac{j_{\rm out}}{j_{\rm in}} = \frac{v|\psi_{\rm out}|^2}{v|\psi_{\rm in}|^2} = \frac{v(X_{\rm out}(b)/b)^2}{v(X_{\rm in}(a)/a)^2} = \frac{a^2}{b^2}\exp\left[-\frac{2\,{\rm Im}\,S_0}{\hbar}\right]. \tag{16}$$

Let us now calculate the phase space factor corresponding to the black hole tunnelling. For a Schwarzschild black hole, the line element in Painlevé coordinates is

$$ds^2 = -c^2\left(1 - \frac{2MG}{c^2r}\right)dt^2 + 2c\sqrt{\frac{2MG}{c^2r}}\,dt\,dr + dr^2$$
$$+ r^2(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{17}$$

and the radial null geodesics are

$$\dot r = \frac{dr}{dt} = \pm c\left(1 - \sqrt{\frac{2MG}{c^2r}}\right), \tag{18}$$

with the upper (lower) sign in Eq. (18) corresponding to outgoing (ingoing) geodesics, under the implicit assumption that t increases towards the future [47].

In this Letter, however, we consider the tunneling of a massive particle. That is, the outgoing particle is a massive shell (de Broglie S-wave). Such massive quanta do not follow radial-lightlike geodesics (18). In analogy to Ref. [19], we treat the massive particle as a de Broglie wave and obtain the expression for $\dot r$. Namely:

$$\dot r = v_p = \frac{1}{2}v_g = -\frac{1}{2}\frac{g_{00}}{g_{01}} = \frac{1}{2r}\frac{c^2r^2 - 2MGr}{\sqrt{2MGr}}. \tag{19}$$

Note that, to calculate the emission rate correctly, we should take into account the self-gravitation of the tunnelling particle, here assumed to have energy ω. That is, we should replace M with $M - \omega$ in (17) and (19) to describe the motion of the particle correctly [1–3].

The canonical momentum p_r and the imaginary part of the action ${\rm Im}\,S_0$ can be easily obtained. Namely:

$$p_r = \int_0^{p_r} dp_r' = \int\frac{dH}{\dot r} = -i\pi\frac{\hbar}{l_p^2}r, \tag{20}$$

$${\rm Im}\,S_0 = {\rm Im}\int_{r_i}^{r_f} p_r\,dr = -\frac{1}{2}\hbar\left[\frac{A_f}{4l_p^2} - \frac{A_i}{4l_p^2}\right]. \tag{21}$$

The probability of barrier penetration is

$$\Gamma_p = \frac{r_i^2}{r_f^2}\exp\left[-\frac{2\,{\rm Im}\,S_0}{\hbar}\right]$$
$$= \exp\left[\left(\frac{A_f}{4l_p^2} - \ln\frac{A_f}{4l_p^2}\right) - \left(\frac{A_i}{4l_p^2} - \ln\frac{A_i}{4l_p^2}\right)\right], \tag{22}$$

where $l_p^2 = \hbar G/c^3$. In this Letter, we investigate the transition of the matter-gravity system from one spherically symmetric state to another at the same energy. This transition corresponds to the production and barrier penetration of a massive spherical shell (or a massless shell). To be specific, this process proceeds in two