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Stable first-order particle-frame relativistic hydrodynamics for dissipative systems

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ABSTRACT

We propose a stable first-order relativistic dissipative hydrodynamic equation in the particle frame (Eckart frame) for the first time. The equation to be proposed was in fact previously derived by the authors and a collaborator from the relativistic Boltzmann equation. We demonstrate that the equilibrium state is stable with respect to the time evolution described by our hydrodynamic equation in the particle frame. Our equation may be a proper starting point for constructing second-order causal relativistic hydrodynamics, to replace Eckart's particle-flow theory.

Relativistic hydrodynamics (RHD) is a useful tool for analyzing slow and long wavelength behavior of relativistic many-particle systems in terms of static and dynamic thermodynamic properties. In fact, RHD is widely used in astrophysics [1] and the phenomenology of relativistic heavy ion collisions [2]. Since works demonstrating the success of perfect hydrodynamics in describing the phenomenology of the Relativistic Heavy Ion Collider (RHIC) at BNL [2–4], we are witnessing a growing interest in RHD for *dissipative* systems [5–8]. Indeed, there have been many works attempting to show how small can be the transport coefficients of strongly-interacting systems composed of hadrons or quarks and gluons, with many of these employing the so-called AdS/CFT correspondence hypothesis [9]. It should be noticed, however, that the theory of RHD for dissipative systems is not clearly established, although there have been many fundamental studies since Eckart's pioneering work [10].

We identify the following three fundamental problems regarding relativistic hydrodynamic equations (RHDEs) for dissipative fluids [11]: (a) ambiguities in the definition of the fluid flow [5,7,10,12–14]; (b) the unphysical instability of the equilibrium state in the theory of the so-called first-order equations, in particular in the Eckart frame [15], defined below; (c) the lack of causality in the first-order equations [14,16–18]. The present Letter is concerned with the first two problems. The unphysical instability of the equilibrium state may be attributable to the lack of causality, and the Israel–Stewart equations with second-order time-derivative are presently being examined in connection to this problem [5–7,18]. In fact, the Israel–Stewart second-order formalism restores the instability of the equilibrium state in the Eckart equation [15]. However, we emphasize that the first two problems and the third one have different origins, and the first two must be resolved before the third is addressed. Note that the causality problem also exists in non-relativistic cases and is in essence a problem of how to incorporate the space–time scales shorter than those corresponding to the mean-free path, beyond those in the usual hydrodynamic regime. We also remark that the proper form of Israel–Stewart-type equations has not yet been definitely determined [5,7].

Let us represent the flow velocity by u^μ, with $u_\mu u^\mu = g^{\mu\nu} u_\mu u_\nu = 1$ ($g^{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$). In the relativistic theory, the rest frame of the fluid and the flow velocity u^μ cannot be uniquely defined when there exist viscosity and heat conduction. In the phenomenological theories [10,12], the ambiguity of the flow velocity u^μ is resolved by placing constraints on the dissipative part of the energy–momentum tensor, $\delta T^{\mu\nu}$, and the particle current, δN^μ. Landau and Lifshitz defined u^μ such that there is no dissipative energy density, energy flow nor particle density; i.e., we have the constraints $\delta T^{\mu\nu} u_\nu = 0$ (referred to as ET) and $u_\mu \delta N^\mu = 0$ (EN). This frame is called the energy frame. Contrastingly, Eckart chose the particle frame, in which there is no dissipative contribution to the particle current; i.e., we have $\delta N^\mu = 0$ (PN), together with $u_\mu u_\nu \delta T^{\mu\nu} = 0$ (PT): These conditions imply that there is no dissipative contribution to the energy density in this frame. However, it should be noted that the seemingly plausible constraint PT on $\delta T^{\mu\nu}$ is problematic, as shown in [11] and explained below.

Recently, Tsumura, Kunihiro (the present authors) and Ohnishi (abbreviated as TKO) [11] derived generic covariant hydrodynamic equations for a viscous fluid through a reduction of the dynamics

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described by the relativistic Boltzmann equation in a systematic manner, with no heuristic arguments, on the basis of the so-called renormalization group (RG) method [19–21]. This was done by introducing the macroscopic frame vector $\boldsymbol{a}^\mu$ that defines the macroscopic Lorenz frame, in which the slow dynamics are described. The generic equation derived by TKO can produce a relativistic dissipative hydrodynamic equation in any frame with the appropriate choice of $\boldsymbol{a}^\mu$; the resulting equation in the energy frame coincides with that of Landau and Lifshitz [12], while that in the particle frame is similar to, but slightly different from, the Eckart equation. Interestingly, the TKO equation in the particle frame does not satisfy the constraints PT on $\delta T^{\mu\nu}$ but, instead, satisfies $\delta T^{\mu}{}_{\mu} = 0$, which we call PT', together with PN. It should be noted that the new constraints, PT', are identical to a matching condition postulated by Marle and Stewart (MS) in the derivation of the RHD from the Boltzmann equation with use of Grad's moment theory [22]. We call the constraints PT', together with PN, the Grad–Marle–Stewart (GMS) constraints. In [11], TKO proved that the simultaneous constraints PT and PN cannot be compatible with the underlying Boltzmann equation if the hydrodynamic equation describes the slow, long wavelength limit of the solutions of the Boltzmann equation. This is interesting in connection to problem (b), i.e., the fact that the solutions of the Eckart equation around the thermal equilibrium are unstable [15], while the Landau theory is stable.

An immediate question is whether the solutions of the new equations in the particle frame are stable around the thermal equilibrium. In fact, the hydrodynamic equations of MS and TKO in the particle frame are of different forms, although both satisfy the constraints PT' and PN. In the present Letter, we examine the stability problem for the new equations in the particle frame. Because second-order equations, such as the Israel–Stewart equations, are usually constructed in the particle frame, as an extension of the Eckart equation, finding a stable first-order equation in the particle frame is of fundamental significance. As the RG method has been employed to construct the slow dynamics of various systems through the explicit construction of the slow, stable manifold of the dynamics, we conjecture that the hydrodynamic equation obtained as the slow, long wavelength limit of the Boltzmann equation on the basis of the RG method will provide a description in which the thermal equilibrium state is stable. We demonstrate that this is indeed the case by performing a linear stability analysis using the EOS and the transport coefficients for a rarefied gas. By contrast, we find that the MS equation, like the Eckart equation, is unstable. Hence, for the first time, a stable RHDE is obtained in the particle frame. We believe that this will provide a sound starting point for the construction of the proper second-order equations.

The energy–momentum tensor for our equation in the particle frame reads

$$\begin{aligned} T^{\mu\nu} &= \epsilon u^\mu u^\nu - p\Delta^{\mu\nu} + \lambda u^\mu \nabla^\nu T + \lambda u^\nu \nabla^\mu T \\ &\quad + \zeta\big(3u^\mu u^\nu - \Delta^{\mu\nu}\big)\big[-(3\gamma-4)^{-2}\nabla\cdot u\big] \\ &\quad + \eta\big(\nabla^\mu u^\nu + \nabla^\nu u^\mu - 2/3\Delta^{\mu\nu}\nabla\cdot u\big), \end{aligned} \tag{1}$$

while the particle current is given by $N^\mu = nu^\mu$, with $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ and $\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$. Here T, μ, ϵ, p, n and γ are the temperature, the chemical potential, the internal energy, the pressure, the particle density and the ratio of the specific heats, respectively, and ζ, λ and η denote the bulk viscosity, the heat conductivity and the shear viscosity, respectively. The MS equations are obtained from the above equations through the replacements $-\zeta(3\gamma-4)^{-2}\nabla\cdot u \to +\zeta(3\gamma-4)^{-1}\nabla\cdot u$ and $\lambda\nabla^\mu T \to \lambda(\nabla^\mu T - TDu^\mu)$, where $D \equiv u^\nu\partial_\nu$. One can easily check that both equations satisfy the GMS constraints. Nevertheless, we find the following differences between them: (A) the thermal forces in the MS equations contain the time-like derivative of the flow velocity Du^μ, while those in our equations involve only the space-like derivative ∇^μ, and (B) the sign of the thermodynamic force owing to the bulk viscosity in our equation is the same as that in the Landau equation and opposite that in the MS equation. We can trace the two characteristic features of our theory back to the simple ansatz that only the spatial inhomogeneity, over distances of the order of the mean free path, is the origin of the dissipation. It should be noted that the same ansatz for the non-relativistic case leads naturally to the Navier–Stokes equation, as shown in [21], and hence our framework can be interpreted as the most natural covariantization of the non-relativistic case.

The thermal equilibrium state is given by $u^\mu(x) = (1,0,0,0) \equiv u_0^\mu$, $T(x) = T_0$ and $\mu(x) = \mu_0$, with T_0 and μ_0 being constant. This is a trivial solution to the equations. Let us investigate the linear stability of the equilibrium solution. Writing $T(x) = T_0 + \delta T(x)$, $\mu(x) = \mu_0 + \delta\mu(x)$ and $u^\mu(x) = u_0^\mu + \delta u^\mu(x)$, we examine the time evolution of the deviations in the linear approximation using the evolution equation given by $\partial_\mu T^{\mu\nu} = 0$ and $\partial_\mu N^\mu = 0$. Here we note that the independent variables are the five quantities $\delta T(x)$, $\delta\mu(x)$ and $\delta u^i(x)$ $(i = 1,2,3)$, because $\delta u^0(x) = 0$, due to the constraint $u_\mu(x)u^\mu(x) = 1$.

In terms of the Fourier components $\tilde{\Phi}_\alpha(k) \equiv {}^{\mathrm{t}}(\delta\tilde{u}^1(k), \delta\tilde{u}^2(k), \delta\tilde{u}^3(k), \delta\tilde{T}(k), \delta\tilde{\mu}(k))$, defined through $\Phi_\alpha(x) = \int \frac{d^4k}{(2\pi)^4}\tilde{\Phi}_\alpha(k)e^{-ik\cdot x}$, the linearized hydrodynamic equation reduces to the algebraic equation $\sum_{\beta=1}^5 M_{\alpha\beta}\tilde{\Phi}_\beta = 0$, with

$$M_{\alpha\beta} \equiv \begin{pmatrix} \mathcal{L}_1 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{L}_1 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{L}_1 - \mathcal{L}_2(k^3)^2 & i\mathcal{L}_3k^3 & i\mathcal{L}_4k^3 \\ 0 & 0 & -i\mathcal{L}_5k^3 & \mathcal{L}_6 & \mathcal{L}_7 \\ 0 & 0 & -i\mathcal{L}_8k^3 & \mathcal{L}_9 & \mathcal{L}_{10} \end{pmatrix}, \tag{2}$$

where we have set $k^\mu = (k^0, 0, 0, k^3)$ without loss of generality. The first and second components of $\tilde{\Phi}_\alpha$ describe the transverse mode, while the third component the longitudinal one. Here $\mathcal{L}_{i=1-10}$ are given by $\mathcal{L}_1 \equiv (\epsilon+p)(-ik^0) + \eta|\boldsymbol{k}|^2$, $\mathcal{L}_2 \equiv -\eta/3 - \zeta_P$, $\mathcal{L}_3 \equiv \partial p/\partial T - \lambda(-ik^0)$, $\mathcal{L}_4 \equiv \partial p/\partial\mu$, $\mathcal{L}_5 \equiv -(\epsilon+p) + 3\zeta_P(-ik^0)$ $\mathcal{L}_6 \equiv \partial\epsilon/\partial T(-ik^0) + \lambda|\boldsymbol{k}|^2$, $\mathcal{L}_7 \equiv \partial\epsilon/\partial\mu(-ik^0)$, $\mathcal{L}_8 \equiv -n$, $\mathcal{L}_9 \equiv \partial n/\partial T(-ik^0)$ and $\mathcal{L}_{10} \equiv \partial n/\partial\mu(-ik^0)$, with $\zeta_P \equiv \zeta(3\gamma-4)^{-2}$ being the effective bulk viscosity in the particle frame. In the above, the quantities, ϵ, p, n, γ, ζ, λ, η, $\partial\epsilon/\partial T$, $\partial\epsilon/\partial\mu$, $\partial p/\partial T$, $\partial p/\partial\mu$, $\partial n/\partial T$ and $\partial n/\partial\mu$ take their equilibrium values, with $T = T_0$ and $\mu = \mu_0$.

The existence condition of a solution reads $\det M = 0$, which reduces to

$$\begin{aligned} &\mathcal{L}_1^2\big[\big(\mathcal{L}_1 - |\boldsymbol{k}|^2\mathcal{L}_2\big)(\mathcal{L}_6\mathcal{L}_{10} - \mathcal{L}_7\mathcal{L}_9) - |\boldsymbol{k}|^2\mathcal{L}_5(\mathcal{L}_3\mathcal{L}_{10} - \mathcal{L}_4\mathcal{L}_9) \\ &\quad - |\boldsymbol{k}|^2\mathcal{L}_8(\mathcal{L}_4\mathcal{L}_6 - \mathcal{L}_3\mathcal{L}_7)\big] = 0. \end{aligned} \tag{3}$$

This equation gives the dispersion relation $k^0 = k^0(|\boldsymbol{k}|)$ for the hydrodynamic modes, and the stability condition for the equilibrium state reads $\operatorname{Im} k^0 \leqslant 0$, $\forall|\boldsymbol{k}|$.

We see the dispersion relation for the transverse mode is given by $\mathcal{L}_1 = 0$, whose solution is $k^0 = -i\eta|\boldsymbol{k}|^2/(\epsilon+p)$. Thus, we find that the transverse mode is stable.

Here we again stress that the equation we study does not contain a term proportional to Du^μ in the thermal force for the heat flow. What would happen if such a term were present in the thermal forces, as in the case of the MS and the Eckart theories? In this case, the corresponding equation becomes $\mathcal{L}_1 = (\epsilon+p)(-ik^0) - T\lambda(-ik^0)^2 + \eta|\boldsymbol{k}|^2 = 0$, which possesses a root with a positive imaginary part, and hence an unstable transverse mode appears. We emphasize that this instability is inevitable if the heat flow term contains Du^μ [8].