Physics Letters B 668 (2008) 442–446

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

On Born's deformed reciprocal complex gravitational theory and noncommutative gravity

Carlos Castro

Center for Theoretical Studies of Physical Systems, Clark Atlanta University, Atlanta, GA 30314, USA

ARTICLE INFO

Article history:
Received 13 August 2008
Received in revised form 24 August 2008
Accepted 27 August 2008
Available online 4 September 2008
Editor: M. Cvetič

ABSTRACT

Born's reciprocal relativity in flat spacetimes is based on the principle of a *maximal* speed limit (speed of light) and a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality) and where coordinates and momenta are unified on a single footing. We extend Born's theory to the case of curved spacetimes and construct a *deformed* Born reciprocal general relativity theory in curved spacetimes (without the need to introduce star products) as a local gauge theory of the *deformed* Quaplectic group that is given by the semi-direct product of $U(1,3)$ with the *deformed* (noncommutative) Weyl–Heisenberg group corresponding to *noncommutative* generators $[Z_a, Z_b] \neq 0$. The Hermitian metric is complex-valued with symmetric and nonsymmetric components and there are *two* different complex-valued Hermitian Ricci tensors $\mathcal{R}_{\mu\nu}, \mathcal{S}_{\mu\nu}$. The deformed Born's reciprocal gravitational action linear in the Ricci scalars $\mathcal{R}, \mathcal{S}$ with Torsion-squared terms and BF terms is presented. The plausible interpretation of $Z_\mu = E^a_\mu Z_a$ as *noncommuting* p-brane background complex spacetime coordinates is discussed in the conclusion, where E^a_μ is the complex vielbein associated with the Hermitian metric $G_{\mu\nu} = g_{(\mu\nu)} + ig_{[\mu\nu]} = E^a_\mu \bar{E}^b_\nu \eta_{ab}$. This could be one of the underlying reasons why string-theory involves gravity.

Born's reciprocal ("dual") relativity [1] was proposed long ago based on the idea that coordinates and momenta should be unified on the same footing, and consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. A *curved* phase space case scenario has been analyzed by Brandt [2] within the context of the Finsler geometry of the $8D$ tangent bundle of spacetime where there is a limiting value to the proper acceleration and such that generalized $8D$ gravitational equations reduce to ordinary Einstein–Riemannian gravitational equations in the *infinite* acceleration limit. Other relevant work on the principle of maximal acceleration can be found in [3]. For a recent monograph on Finsler geometry see Vacaru [4].

Born's reciprocal "duality" principle is nothing but a manifestation of the large/small tension duality principle reminiscent of the T-duality symmetry in string theory; i.e., namely, a small/large radius duality, a winding modes/Kaluza–Klein modes duality symmetry in string compactifications and the Ultraviolet/Infrared entanglement in noncommutative field theories. Hence, Born's duality principle in exchanging coordinates for momenta could be the underlying physical reason behind T-duality in string theory. The generalized velocity and acceleration boosts (rotations) transformations of the $8D$ Phase space, where X, T, E, P are *all* boosted (rotated) into each-other, were given by [5] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$. It was found later on [6] that Planck scale Areas are Invariant under pure acceleration boosts which may be relevant to string theory.

Invariant actions for a point-particle in reciprocal Relativity involving Casimir group invariant quantities can be found in [7]. Casimir invariant field equations; unitary irreducible representations based on Mackey's theory of induced representations; the relativistic harmonic oscillator and coherent states can be found in [5]. The granular cellular structure of spacetime, the Schrödinger–Robertson inequality, multi-mode squeezed states, a "noncommutative" relativistic phase space geometry, in which position and momentum are interchangeable and frame-dependent was studied by [8]. Born's reciprocity principle in atomic physics and galactic motion based on $(1/r) + (b/p)$ potentials was studied recently by [9] with little effect on atomic physics but with relevant effects on galactic rotation without invoking dark matter.

In this Letter we construct a local gauge theory of the *deformed* (noncommutative) Quaplectic group given by the semidirect product of $U(1,3)$ with the deformed (noncommutative) Weyl–Heisenberg group. The $U(1,3)$ arises as the group that leaves invariant the interval in $8D$ phase $dx_\mu\, dx^\mu + dp_\mu\, dp^\mu$ space, as well as invariant the symplectic two-form $\omega = \omega_{\mu\nu}\, dx^\mu \wedge dp^\nu$, simultaneously. The novel result in this Letter is the *modification* of the Weyl–Heisenberg algebra, not unlike Yang's noncommutative phase space algebra [10].

E-mail address: czarlosromanov@yahoo.com.

0370-2693/$ – see front matter
doi:10.1016/j.physletb.2008.08.066

The deformed Weyl–Heisenberg algebra involves the generators

$$Z_a = \frac{1}{\sqrt{2}}\left(\frac{X_a}{\lambda_l} - i\frac{P_a}{\lambda_p}\right),$$
$$\bar{Z}_a = \frac{1}{\sqrt{2}}\left(\frac{X_a}{\lambda_l} + i\frac{P_a}{\lambda_p}\right), \quad a = 1,2,3,4. \tag{1}$$

Notice that we must *not* confuse the *generators* X_a, P_a (associated with the fiber coordinates of the internal space of the fiber bundle) with the ordinary base spacetime coordinates and momenta x_μ, p_μ. The gauge theory is constructed in the fiber bundle over the base manifold which is a $4D$ curved spacetime with *commuting* coordinates $x^\mu = x^0, x^1, x^2, x^3$. The (deformed) Quaplectic group acts as the automorphism group along the internal fiber coordinates. Therefore we must *not* confuse the *deformed* complex gravity constructed here with the noncommutative gravity work in the literature [11] where the spacetime coordinates x^μ are not commuting.

The four fundamental length, momentum, temporal and energy scales are respectively

$$\lambda_l = \sqrt{\frac{\hbar c}{b}}, \qquad \lambda_p = \sqrt{\frac{\hbar b}{c}}, \qquad \lambda_t = \sqrt{\frac{\hbar}{bc}}, \qquad \lambda_e = \sqrt{\hbar bc}, \tag{2}$$

where b is the *maximal* proper force associated with the Born's reciprocal relativity theory. In the natural units $\hbar = c = b = 1$ all four scales become *unity*. The gravitational coupling is given by

$$G = \frac{c^4}{\mathcal{F}_{\max}} = \frac{c^4}{b} \tag{3}$$

and the four scales coincide then with the Planck length, momentum, time and energy, respectively and we can verify that

$$\mathcal{F}_{\max} = m_P\frac{c^2}{L_P} \sim M_{\text{Universe}}\frac{c^2}{R_H}. \tag{4}$$

It was proposed in [6] that a certain large (Hubble)/small (Planck) scale *duality* was operating in this Born's reciprocal relativity theory reminiscent of the T-duality in string theory compactifications. The Hermitian generators $Z_{ab}, Z_a, \bar{Z}_a, I$ of the $U(1,3)$ algebra and the *deformed* Weyl–Heisenberg algebra obey the relations

$$(Z_{ab})^\dagger = Z_{ab}, \qquad (Z_a)^\dagger = \bar{Z}_a, \qquad I^\dagger = I, \quad a,b = 1,2,3,4. \tag{5}$$

The standard Quaplectic group [5] is given by the semi-direct product of the $U(1,3)$ group and the unmodified Weyl–Heisenberg $H(1,3)$ group: $\mathcal{Q}(1,3) \equiv U(1,3) \otimes_s H(1,3)$ and is defined in terms of the generators $Z_{ab}, Z_a, \bar{Z}_a, I$ with $a,b = 1,2,3,4$.

A careful analysis reveals that the complex generators $Z_a, \bar{Z}_a$ (with Hermitian *and* anti-Hermitian pieces) of the *deformed* Weyl–Heisenberg algebra can be defined in terms of the Hermitian $U(1,4)$ algebra generators Z_{AB}, where $A,B = 1,2,3,4,5$, $a,b = 1,2,3,4$, $\eta_{AB} = \operatorname{diag}(+,-,-,-,-)$, as follows:

$$Z_a = (-i)^{1/2}(Z_{a5} - iZ_{5a}),$$
$$\bar{Z}_a = (i)^{1/2}(Z_{a5} + iZ_{5a}), \qquad Z_{55} = \frac{\mathcal{I}}{2}, \tag{6}$$

the Hermitian generators are $Z_{AB} \equiv \mathcal{E}_A^B$ and $Z_{BA} \equiv \mathcal{E}_B^A$; notice that the position of the indices is very relevant because $Z_{AB} \neq Z_{BA}$. The commutators are

$$[\mathcal{E}_a^b, \mathcal{E}_c^d] = -i\delta_c^b\mathcal{E}_a^d + i\delta_a^d\mathcal{E}_c^b,$$
$$[\mathcal{E}_c^d, \mathcal{E}_a^5] = -i\delta_a^d\mathcal{E}_c^5, \qquad [\mathcal{E}_c^d, \mathcal{E}_5^a] = i\delta_c^a\mathcal{E}_5^d \tag{7}$$

and $[\mathcal{E}_5^5, \mathcal{E}_5^a] = -i\delta_5^5\mathcal{E}_5^a \cdots$ such that now $\mathcal{I}(=2Z_{55})$ *no* longer commutes with $Z_a, \bar{Z}_a$. The generators Z_{ab} of the $U(1,3)$ algebra can be decomposed into the Lorentz-subalgebra generators L_{ab} and the "shear"-like generators M_{ab} as

$$Z_{ab} \equiv \frac{1}{2}(M_{ab} - iL_{ab}), \qquad L_{ab} = L_{[ab]} = i(Z_{ab} - Z_{ba}),$$
$$M_{ab} = M_{(ab)} = (Z_{ab} + Z_{ba}), \tag{8}$$

one can see that the "shear"-like generators M_{ab} are *Hermitian* and the Lorentz generators L_{ab} are *anti-Hermitian* with respect to the fiber internal space indices. The explicit commutation relations of the Hermitian generators Z_{ab} can be rewritten as

$$[L_{ab}, L_{cd}] = (\eta_{bc}L_{ad} - \eta_{ac}L_{bd} - \eta_{bd}L_{ac} + \eta_{ad}L_{bc}), \tag{9a}$$
$$[M_{ab}, M_{cd}] = -(\eta_{bc}L_{ad} + \eta_{ac}L_{bd} + \eta_{bd}L_{ac} + \eta_{ad}L_{bc}), \tag{9b}$$
$$[L_{ab}, M_{cd}] = (\eta_{bc}M_{ad} - \eta_{ac}M_{bd} + \eta_{bd}M_{ac} - \eta_{ad}M_{bc}). \tag{9c}$$

Defining $Z_{ab} = \frac{1}{2}(M_{ab} - iL_{ab})$, $Z_{cd} = \frac{1}{2}(M_{cd} - iL_{cd})$ after straightforward algebra it leads to the $U(3,1)$ commutators

$$[Z_{ab}, Z_{cd}] = -i(\eta_{bc}Z_{ad} - \eta_{ad}Z_{cb}) \tag{9d}$$

as expected, and which requires that the commutators $[M,M] \sim L$ otherwise one would not obtain the $U(3,1)$ commutation relations (9d) nor the Jacobi identities will be satisfied.[1] The commutators of the (anti-Hermitian) Lorentz boosts generators L_{ab} with the X_c, P_c generators are

$$[L_{ab}, X_c] = (\eta_{bc}X_a - \eta_{ac}X_b),$$
$$[L_{ab}, P_c] = (\eta_{bc}P_a - \eta_{ac}P_b). \tag{10a}$$

Since the Hermitian M_{ab} generators are the *reciprocal* boosts transformations which *exchange* X for P, in addition to boosting (rotating) those variables, one has in

$$\left[M_{ab}, \frac{X_c}{\lambda_l}\right] = -\frac{i}{\lambda_p}(\eta_{bc}P_a + \eta_{ac}P_b),$$
$$\left[M_{ab}, \frac{P_c}{\lambda_p}\right] = -\frac{i}{\lambda_l}(\eta_{bc}X_a + \eta_{ac}X_b), \tag{10b}$$

such that upon recurring to Eqs. (6), (7) and/or Eqs. (10) after lowering indices it leads to[2]

$$[Z_{ab}, Z_c] = -\frac{i}{2}\eta_{bc}Z_a + \frac{i}{2}\eta_{ac}Z_b - \frac{1}{2}\eta_{bc}\bar{Z}_a - \frac{1}{2}\eta_{ac}\bar{Z}_b,$$
$$[Z_{ab}, \bar{Z}_c] = -\frac{i}{2}\eta_{bc}\bar{Z}_a + \frac{i}{2}\eta_{ac}\bar{Z}_b + \frac{1}{2}\eta_{bc}Z_a + \frac{1}{2}\eta_{ac}Z_b. \tag{10c}$$

In the noncommutative Yang's phase-space algebra case [10], associated with a noncommutative phase space involving noncommuting spacetime coordinates and momentum x^μ, p^μ, the generator $\mathcal{N}$ which appears in the modified $[x^\mu, p^\nu] = i\hbar\eta^{\mu\nu}\mathcal{N}$ commutator is the *exchange* operator $x \leftrightarrow p$, $[p^\mu, \mathcal{N}] = i\hbar x^\mu/R_H^2$ and $[x^\mu, \mathcal{N}] = iL_P^2 p^\mu/\hbar$. L_P, R_H are taken to be the minimal Planck and maximal Hubble length scales, respectively. The Hubble upper scale R_H corresponds to a *minimal* momentum $\hbar/R_H$, because by "duality" if there is a minimal length there should be a minimal momentum also.

Yang's [10] noncommutative phase space algebra is isomorphic to the conformal algebra $SO(4,2) \sim SU(2,2)$ after the correspondence $x^\mu \leftrightarrow L^{\mu 5}$, $p^\mu \leftrightarrow L^{\mu 6}$, and $\mathcal{N} \leftrightarrow L^{56}$. In the *deformed* Quaplectic algebra case, it is in addition to the $\mathcal{I}$ generator, the M_{ab} generator which plays the role of the exchange operator of X

[1] This corrects a typo in [12].

[2] These commutators *differ* from those in [5] because he chose all generators X, P, M, L to be anti-Hermitian so there are no i terms in the commutators in the r.h.s. of Eq. (10b) and there are also sign changes.