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Natural geometric representation for electron local observables

V.G. Minogin

Institute of Spectroscopy Russ. Ac. of Sci., Fizicheskaya st. 5, 142190, Troitsk, Moscow, Russian Federation

HIGHLIGHTS

- Quartic identities that define the orthogonality relations for the electron local observables are found.
- Joint solution of quartic and bilinear identities defines a unique natural representation of the electron local observables.
- Functional dependence of the electron wave functions on the electron local observables is determined.

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ABSTRACT

An existence of the quartic identities for the electron local observables that define orthogonality relations for the 3D quantities quadratic in the electron observables is found. It is shown that the joint solution of the quartic and bilinear identities for the electron observables defines a unique natural representation of the observables. In the natural representation the vector type electron local observables have well-defined fixed positions with respect to a local 3D orthogonal reference frame. It is shown that the natural representation of the electron local observables can be defined in six different forms depending on a choice of the orthogonal unit vectors. The natural representation is used to determine the functional dependence of the electron wave functions on the local observables valid for any shape of the electron wave packet.

1. Introduction

It is well known that the electron wave packet is characterized by the 16 local observables frequently called the bilinear forms (BFs) or bilinear covariants [1–3]. It is also known that the local observables are not independent as they satisfy the bilinear identities coming from reordering of the

E-mail address: minogin@isan.troitsk.ru.

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Dirac partial wave functions [4–17]. From mathematical point of view the bilinear identities also called the Pauli identities or in most general form the Fierz identities are the direct consequences of the completeness of the Dirac matrices. In convenient form all the bilinear identities can be represented by sets of scalar, vector and tensor type equations which include in total 36 one-component identities [17]. Bilinear identities are usually considered as a useful tool for transforming different mathematical expressions involving products of the electron wave functions.

Alongside with bilinear identities the electron local observables satisfy the higher-order identities such as triple or quartic identities. From the formal mathematical point of view the higher-order identities may have very little interest as any dependent quantities. However, from the physical point of view the higher-order identities may exhibit new geometric properties absent in lower-order identities. Qualitatively, one may expect that triple identities cannot give important new geometric relations as they include only 6 wave functions and accordingly may describe only partial symmetry of the electron matter. On the contrary, the quartic identities include 8 wave functions, twice the number of the Dirac functions, and accordingly may reflect the whole symmetry of the electron matter.

In this paper we show the existence of the quartic identities for the electron BFs which describe the orthogonality relations between the three-component electron quantities quadratic in BFs. Specifically we find that for any choice of the electron wave functions three scalar products between the cross products of the three-component BFs are always equal to zero. We show that any of these three zero scalar products can be used to introduce three orthogonal unit vectors constituting a local orthogonal reference frame in 3D. Such a frame can be constructed by 6 different ways and may be considered as a natural local frame for any electron wave packet. The use of such a local frame allows one to decompose all the three-component electron observables over the chosen orthogonal unit vectors. Representation of the electron local observables through the local orthogonal unit vectors also solves the problem of minimal description of the observables as it includes a minimal number of 7 real quantities defined by 4 one-component BFs and 3 orthogonal unit vectors.

We also show that the found natural representation of the electron observables allows one to construct general functional dependence of the electron wave functions on the local observables. In particular, we show that the use of the standard representation of the Dirac matrices gives the explicit structure of the wave functions in the "amplitude–phase" form valid for any shape of the electron wave packet.

2. Wave equation and bilinear forms

In what follows we use the Dirac equation for the four-component electron wave function $\psi = \psi(\mathbf{r}, t)$ in a form explicitly resolved with respect to the time derivative

$$i\partial_0\psi = -i\boldsymbol{\alpha}\cdot\nabla\psi + m\beta\psi, \tag{1}$$

where $\partial_0 = \partial/\partial t$, $\nabla = \partial/\partial\mathbf{r}$, and $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$ and β are the Dirac 4×4 matrices. Eq. (1) is written in units $\hbar = c = 1$. The Dirac matrices are assumed to satisfy usual (anti)commutation relations

$$\alpha^k\alpha^l + \alpha^l\alpha^k = 2\delta_{kl}I, \qquad \boldsymbol{\alpha}\beta + \beta\boldsymbol{\alpha} = 0, \qquad \beta^2 = I, \tag{2}$$

where k, l are the "three-dimensional" indices, $k, l = 1, 2, 3$, and I is a unit 4×4 matrix.

It is convenient to introduce the electron local observables by noting that a complete set of independent 4 by 4 matrices connected with the electron wave function ψ includes the 16 Hermitian matrices μ, the unit matrix I and 15 Dirac matrices $\alpha^1, \alpha^2, \alpha^3, \beta, \ldots$. All these 16 matrices can be obtained by the multiplication of the initial Dirac matrices α^k and β and introducing new Hermitian matrices as shown in the multiplication table of paper [17]. This procedure directly gives the 16 BFs $\langle\mu\rangle = \psi^\dagger\mu\psi$ based on the 16 Hermitian 4×4 matrices $\mu = I, \boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}, \boldsymbol{\delta}, \zeta^0, \boldsymbol{\zeta}, \eta$ where the additional Dirac matrices are

$$\begin{aligned}
&\boldsymbol{\gamma} = (\gamma^1, \gamma^2, \gamma^3) = i(\alpha^2\alpha^3\beta, \alpha^3\alpha^1\beta, \alpha^1\alpha^2\beta),\\
&\boldsymbol{\delta} = (\delta^1, \delta^2, \delta^3) = -i\boldsymbol{\alpha}\beta, \qquad \zeta^0 = i\alpha^1\alpha^2\alpha^3,\\
&\boldsymbol{\zeta} = (\zeta^1, \zeta^2, \zeta^3) = i(\alpha^2\alpha^3, \alpha^3\alpha^1, \alpha^1\alpha^2),\\
&\eta = -i\zeta^0\beta = \alpha^1\alpha^2\alpha^3\beta.
\end{aligned} \tag{3}$$