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Emergent spin

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HIGHLIGHTS

- The spin–statistics theorem is not required for particles on a lattice.
- Spin emerges dynamically when spinless fermions have a relativistic continuum limit.
- Graphene and staggered fermions are examples of this phenomenon.
- The phenomenon is intimately tied to chiral symmetry and fermion doubling.
- Anomaly cancellation is a crucial feature of any valid lattice fermion action.

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ABSTRACT

Quantum mechanics and relativity in the continuum imply the well known spin–statistics connection. However for particles hopping on a lattice, there is no such constraint. If a lattice model yields a relativistic field theory in a continuum limit, this constraint must "emerge" for physical excitations. We discuss a few models where a spin-less fermion hopping on a lattice gives excitations which satisfy the continuum Dirac equation. This includes such well known systems such as graphene and staggered fermions.

1. Introduction

As is well known, in quantum field theory the constraints of quantum mechanics with special relativity give rise to the spin–statistics connection. In particular, fermions must have half integer spin. However in a lattice theory, the lattice structure itself breaks relativity. In a lattice model one is free to formulate a model based on spin-less fermions. If one finds that the low energy excitations of such a model have a relativistic spectrum, then these excitations must carry half integer spin. In some sense spin must "emerge" from the dynamics.

Remarkably such models exist. The most famous is based on graphene. The Hamiltonian for spin-less fermions hopping on a hexagonal two dimensional lattice is easily diagonalized [1], and the low

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energy excitations above the half filled system do indeed mimic a relativistic spectrum. And these excitations satisfy the two dimensional Dirac equation, correspondingly carrying half integer angular momentum [2].

This situation is not unique. Another example, discussed in more detail below, is a two dimensional square lattice subjected to a magnetic field of half a flux unit per elementary square. This model generalizes to more dimensions by threading the magnetic field through all elementary plaquettes. This is a route to the well known staggered fermion theory [3–5].

This paper reviews these models and discusses some of the interesting relations with chiral symmetry and doubling issues. Section 2 goes through the standard solution of the graphene solution in the tight binding limit. Section 3 discusses the close ties between the doubling issues of lattice fermions and topology. Also we see how a chiral symmetry protects masses from an additive renormalization. Section 4 generalizes these ideas to a square lattice in a magnetic field and makes the connection to staggered fermions. Section 5 extends this idea to higher dimensions. Section 6 discusses issues that can arrive in going from the Hamiltonian version of staggered fermions to a Euclidean path integral approach. Section 7 turns to the introduction of gauge fields and an amusing property when the gauge group is $SU(N)$ with N even. In Section 8 we make some general observations on the effects of gauge field topology on the lattice fermion spectrum. In particular we present a variation of the Nielsen–Ninomiya theorem [6] that applies to all lattice actions including mass terms. Finally there are some brief conclusions in Section 9.

2. Graphene

As is well known, the solution to a theory of fermions hopping on a hexagonal lattice displays two Dirac cones. With small excitations around half filling, each of these cones represents a fermion satisfying the Dirac equation. Graphene, basically a two dimensional hexagonal planar structure, represents a realization of this system [1].

In the physical situation the electrons already have spin, so the extra doubling of species with the two cones can be thought of as representing "flavor" or "isospin". However, from a theoretical point of view one can consider spin-less fermions hopping on the lattice, and then the excitations will formally acquire spin one-half. In this section we review this solution.

To solve this problem, it is useful to use a fortuitous set of coordinates, as sketched in Fig. 1. Orienting the lattice as in the figure, the sites can be considered as being of two types. We consider type a sites on the left side of each horizontal bond, and b sites on the right. These sites are labeled with non-orthogonal coordinates x_1 and x_2 labeling the horizontal bonds. The two axes are not orthogonal, but oriented at 30° from the horizontal. Associated with each site is a pair of creation–annihilation operators, labeled $(a^\dagger, a)$ and $(b^\dagger, b)$ respectively. These satisfy the usual anti-commutation relations

$$\begin{aligned} &[a_{x_1,x_2}, a^\dagger_{y_1,y_2}]_+ = \delta_{x_1,y_1}\delta_{x_2,y_2} \\ &[a_{x_1,x_2}, b^\dagger_{y_1,y_2}]_+ = [b_{x_1,x_2}, a^\dagger_{y_1,y_2}]_+ = 0. \end{aligned} \tag{1}$$

With these coordinates, the nearest neighbor Hamiltonian takes the form

$$\begin{aligned} H = K \sum_{x_1,x_2} & a^\dagger_{x_1,x_2} b_{x_1,x_2} + b^\dagger_{x_1,x_2} a_{x_1,x_2} + a^\dagger_{x_1+1,x_2} b_{x_1,x_2} + b^\dagger_{x_1,x_2} a_{x_1+1,x_2} \\ & + a^\dagger_{x_1,x_2} b_{x_1,x_2+1} + b^\dagger_{x_1,x_2+1} a_{x_1,x_2}. \end{aligned} \tag{2}$$

The three terms correspond to horizontal, upward to the right, and upward to the left bonds respectively. Here K is usually referred to as the "hopping" parameter, and sets the energy scale.

To solve this system, go to momentum (reciprocal) space and define

$$\tilde{a}_{p_1,p_2} = \sum_{x_1,x_2} e^{-ip_1x_1}\, e^{-ip_2x_2}\, a_{x_1,x_2}. \tag{3}$$

These satisfy the commutation relations

$$[\tilde{a}_{p'_1,p'_2}, \tilde{a}_{p_1,p_2}]_+ = (2\pi)^2 \delta(p'_1, p_1)\delta(p'_2, p_2). \tag{4}$$