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Distributions of off-diagonal scattering matrix elements: Exact results

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HIGHLIGHTS

- Scattering problem in complex or chaotic systems.
- Heidelberg approach to model the chaotic nature of the scattering center.
- A novel route to the nonlinear sigma model based on the characteristic function.
- Exact results for the distributions of off-diagonal scattering-matrix elements.
- Universal aspects of the scattering-matrix fluctuations.

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ABSTRACT

Scattering is a ubiquitous phenomenon which is observed in a variety of physical systems which span a wide range of length scales. The scattering matrix is the key quantity which provides a complete description of the scattering process. The universal features of scattering in chaotic systems is most generally modeled by the Heidelberg approach which introduces stochasticity to the scattering matrix at the level of the Hamiltonian describing the scattering center. The statistics of the scattering matrix is obtained by averaging over the ensemble of random Hamiltonians of appropriate symmetry. We derive exact results for the distributions of the real and imaginary parts of the off-diagonal scattering matrix elements applicable to orthogonally-invariant and unitarily-invariant Hamiltonians, thereby solving a long standing problem.

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1. Introduction

Scattering is a truly fundamental issue in physics [1,2]. A major part of our information about quantum systems stems from scattering experiments. Rutherford's gold-foil experiment [3] is a classic example which led us towards the understanding of the atomic structure. Even in modern times, powerful particle accelerators rely on scattering experiments to probe deeper and deeper into the structure of matter. Moreover, scattering plays a crucial role in classical wave systems as well and one can often relate the relevant observables to the scattering parameters. Along with the atomic nuclei [4–8], atoms [9–12] and molecules [13–15], some of the other important examples where scattering phenomena have been of considerable interest are mesoscopic ballistic devices [16–27], microwave cavities [28–43], irregular graphs [44,45], quantum graphs [46–48], elastomechanical billiards [49–51], wireless communication [52–54] etc.

The scattering process can be completely described in terms of the scattering matrix (S matrix). It relates the asymptotic initial and final Hilbert spaces spanned by a quantum system undergoing the scattering process. In simple words, it relates the incoming and outgoing waves. In a quantum mechanical context these are the wave functions, i.e. the probability amplitudes. However, in classical systems, the waves are the displacement vectors in elastomechanical systems or the electromagnetic field in microwave cavities. The flux conservation requirement constrains the S matrix to be unitary, i.e., $SS^\dagger = S^\dagger S = \mathbb{1}$. As a consequence of the complicated dependence on the parameters of the incoming waves and the scattering center, scattering is quite often of chaotic nature. Accordingly one needs a statistical description of the scattering phenomenon and hence of the S matrix, i.e., to describe the S matrix and related observables in terms of correlations functions and distributions. Two standard approaches in this direction are the semiclassical approach [55–58] and the stochastic approach [59–62]. The former relies on representing the S-matrix elements in terms of a sum over the classical periodic orbits, starting with the genuine microscopic Hamiltonian representing the system. The latter, in contrast, relies on introducing stochasticity to the scattering matrix or to the Hamiltonian describing the scattering center. Both of these have their advantages and drawbacks. For instance, the semiclassical approach suffers the restriction caused by an exponential proliferation of classical periodic unstable trajectories. It is further constrained by the formal condition $\hbar \to 0$ which demands that the number of open channels be large and therefore does not cover all interesting cases. The stochastic approach, on the other hand, gets restricted by the very nature of the stochastic modeling. Moreover, in this case, one can expect only to explore the universal aspects, leaving aside the system specific properties. The comparison between these two approaches has been discussed in detail in [63].

As indicated above, within the stochastic approach, one can pursue one of the following two routes. In the first one, the S matrix itself is regarded a stochastic quantity and is described by the Poisson kernel. Its derivation is based on imposing minimal information content along with the necessary conditions like unitarity, analyticity etc. This route was pioneered by Mello and coworkers and is often referred to as the Mexico approach [61,62]. The second path relies on introducing the stochasticity at the level of the Hamiltonian describing the scattering center. For this, one employs the random matrix universality conjecture and models the system Hamiltonian by one of the appropriate random matrix ensembles [64–66]. This path was laid by Weidenmüller and coworkers [59] and is referred to as the Heidelberg approach. Even though these two stochastic approaches appear very different in their formulation, they describe precisely the same quantity, the S matrix. Naturally, one would expect that these two routes are equivalent. Indeed it was shown by Brouwer that the Poisson kernel can be derived using the Heidelberg approach by modeling the scattering-center Hamiltonian by a Lorentzian (or Cauchy) ensemble of random matrices [67]. Since the universal properties depend only on the invariance properties of the underlying Hamiltonian [64–66], his result established the equivalence between the two approaches. Furthermore, very recently Fyodorov et al. have demonstrated this equivalence for a broad class of unitary-invariant ensembles of random matrices [68].

In their pioneering work Verbaarschot et al. [69] calculated the two-point energy correlation functions by implementing the supersymmetry technique [70–73] within the Heidelberg approach. Their result established the universality of the S-matrix fluctuation properties in chaotic scattering. Further progress in characterizing the S-matrix fluctuations was made in [74,75] where the authors