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Non-Hermitian Hamiltonians with unitary and antiunitary symmetries

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HIGHLIGHTS

- PT-symmetric Hamiltonians exhibit real eigenvalues when PT symmetry is unbroken.
- PT-symmetric multidimensional oscillators appear to show PT phase transitions.
- This transition was conjectured to be a high-energy phenomenon.
- We show that point group symmetry is useful for predicting broken PT symmetry in multidimensional oscillators.
- PT-symmetric oscillators with C_{2v} symmetry exhibit phase transitions at the trivial Hermitian limit.

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ABSTRACT

We analyse several non-Hermitian Hamiltonians with antiunitary symmetry from the point of view of their point-group symmetry. It enables us to predict the degeneracy of the energy levels and to reduce the dimension of the matrices necessary for the diagonalization of the Hamiltonian in a given basis set. We can also classify the solutions according to the irreducible representations of the point group and thus analyse their properties separately. One of the main results of this paper is that some PT-symmetric Hamiltonians with point-group symmetry C_{2v} exhibit complex eigenvalues for all values of a potential parameter. In such cases the PT phase transition takes place at the trivial Hermitian limit which suggests that the phenomenon is not robust. Point-group symmetry enables us to explain such anomalous behaviour and to choose a suitable antiunitary operator for the PT symmetry.

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1. Introduction

It was shown some time ago that some complex non-Hermitian Hamiltonians may exhibit real eigenvalues [1,2]. The conjecture that such intriguing feature may be due to unbroken PT-symmetry [3] gave rise to a very active field of research [4] (and references therein). The first studied PT-symmetric models were mainly one-dimensional anharmonic oscillators [3–6] and lately the focus shifted towards multidimensional problems [7–15]. Among the most widely studied multidimensional PT-symmetric models we mention the complex versions of the Barbanis [7,8,10–12,14,15] and Hénon–Heiles [7,12] Hamiltonians. Several methods have been applied to the calculation of their spectra: the diagonalization method [7–10,12,14], perturbation theory [7,9,10,12], classical and semiclassical approaches [7,8], among others [12,15]. Typically, those models depend on a potential parameter g so that the Hamiltonian is Hermitian when $g = 0$ and non-Hermitian when $g \neq 0$. Bender and Weir [14] conjectured that the models studied so far may exhibit PT phase transitions so that their spectra are entirely real for sufficiently small but nonzero values of $|g|$. Such phase transition appears to be a high-energy phenomenon.

Multidimensional oscillators exhibit point-group symmetry (PGS) [16,17]. As far as we know such a property has not been taken into consideration in those earlier studies of the PT-symmetric models, except for the occasional parity in one of the variables. It is to be expected that PGS may be relevant to the study of the spectra of multidimensional PT-symmetric anharmonic oscillators. One of the purposes of this paper is to start such research.

The main interest in the study of PT-symmetric oscillators has been to enlarge the class of such models that exhibit real spectra, at least for some values of the potential parameter. In such cases PT-symmetry is broken at particular values $g = g_c$ of the parameter that are known as exceptional points [18–21] and can be easily calculated as critical parameters by means of the diagonalization method [22]. The PT phase transition is determined by the smallest $|g_c|$. Another goal of this paper is to test that conjecture about PT phase transitions by trying to find PT-symmetric models that do not exhibit real spectra, except at the trivial Hermitian limit $g = 0$.

In Section 2 we outline the main ideas of unitary (point-group) and antiunitary symmetries. In Section 3 we show that two exactly solvable PT-symmetric oscillators with different PGS exhibit quite different spectra. One of them shows a phase transition at the trivial Hermitian limit. In Section 4 we discuss some non-Hermitian operators, already studied earlier by other authors, from the point of view of PGS. All of them have been shown to exhibit nontrivial phase transitions. In Section 5 we show a PT-symmetric anharmonic oscillator with complex eigenvalues for all values of the potential parameter. In Section 6 we explain why the PT symmetry is broken for the models in Sections 3 and 5. Finally, in Section 7 we summarize the main results of the paper and draw conclusions.

2. Unitary and antiunitary symmetries

We assume that there is a group of unitary transformations $G = \{U_1, U_2, \ldots, U_n\}$ and a set of antiunitary transformations $S = \{A_1, A_2, \ldots, A_m\}$ that leave the non-Hermitian Hamiltonian operator invariant

$$U_j H U_j^{-1} = H, \qquad A_k H A_k^{-1} = H, \quad j = 1, 2, \ldots, n,\ k = 1, 2, \ldots, m. \tag{1}$$

Therefore, if ψ is an eigenvector of H with eigenvalue E we have

$$HU_j\psi = EU_j\psi, \quad j = 1, 2, \ldots, n \tag{2}$$

and

$$HA_k\psi = E^* A_k\psi, \quad k = 1, 2, \ldots, m. \tag{3}$$

The latter equation tells us that the eigenvalues of H are either real or appear as pairs of conjugate complex numbers.

It is well known that a product of antiunitary operators is a unitary one [23]. Therefore, since A_iA_j leaves the Hamiltonian invariant then $A_iA_j = U_k \in G$, provided that G is the actual symmetry point group for H [24,25].