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Trigonometrical sums connected with the chiral Potts model, Verlinde dimension formula, two-dimensional resistor network, and number theory

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HIGHLIGHTS

- Alternative derivation of certain trigonometrical sums of the chiral Potts model are given.
- Generalization of these trigonometrical sums satisfy recursion formulas.
- The dimension of the space of conformal blocks may be computed from these recursions.
- Exact corner-to-corner resistance, the Kirchhoff index of $2 \times N$ are given.

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ABSTRACT

We have recently developed methods for obtaining exact two-point resistance of the complete graph minus N edges. We use these methods to obtain closed formulas of certain trigonometrical sums that arise in connection with one-dimensional lattice, in proving Scott's conjecture on permanent of Cauchy matrix, and in the perturbative chiral Potts model. The generalized trigonometrical sums of the chiral Potts model are shown to satisfy recursion formulas that are transparent and direct, and differ from those of Gervois and Mehta. By making a change of variables in these recursion formulas, the dimension of the space of conformal blocks of $SU(2)$ and $SO(3)$ WZW models may be computed recursively. Our methods are then extended to compute the corner-to-corner resistance, and the Kirchhoff index of the first non-trivial two-dimensional resistor network, $2 \times N$. Finally, we obtain new closed formulas for variant of trigonometrical sums, some of which appear in connection with number theory.

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1. Introduction

We have recently developed methods for obtaining exact two-point resistance of certain circulant graph namely, the complete graph minus N edges [1]. In this paper, using similar techniques and ideas, we consider trigonometrical sums that arise in the computation of the two-point resistance of the finite resistor networks [2], in the work of McCoy and Orrick on the chiral Potts model [3], and in the Verlinde dimension formula of the twisted/untwisted space of conformal blocks of the $SO(3)/SU(2)$ WZW model [4].

Before considering these trigonometrical sums, we test the techniques used in [1], by first deriving Green's function of the one-dimensional lattice graphs with free boundaries, and the two-point resistance of the N-cycle graph [2]. The same techniques is then used to evaluate a trigonometrical sum that played a crucial role to prove R. F. Scott's conjecture on the permanent of the Cauchy matrix [5,6]. Having tested these techniques, an alternative derivation is then given for certain trigonometrical sum that appeared in the perturbative chiral Potts model [3,7].

We have also considered the general case studied by Gervois and Mehta [7], Berndt and Yeap [8], here, our results agree with those in [7]. It turns out that the Verlinde dimension formulas for the untwisted space of conformal blocks, may be obtained simply by summing over certain parameter of a trigonometrical sum considered in [7]. For the twisted space of conformal blocks, however, the parameter is restricted to take some value. It is shown that the dimension of the conformal blocks on a genus $g \geq 2$ Riemann surface may be obtained through a recursion formula that relates different genera. Mathematically speaking, the dimension of the space of conformal blocks is obtained by expanding certain generating function order by order, or using the Hirzebruch–Riemann–Roch theorem [9].

By using the method given in [1], we are able to obtain closed form formula for the two-point resistance of a $2 \times N$ resistor network [10]. In this paper, an exact computation of the corner-to-corner resistance as well as the total effective resistance of a $2 \times N$ will be given. The total effective resistance, also called the Kirchhoff index [11], this is an invariant quantity of the resistor network or graph. Note that, the exact two-point resistance of an $M \times N$ resistor network is given in terms of a double sum and not in a closed form [2]. Therefore, our computation carried out in this paper, represents the first non-trivial exact results for the two-point resistance of a two-dimensional resistor network.

Having checked that our method works, we used it to evaluate variants of trigonometrical sums, some of which are related to number theory, we hope that these trigonometrical sums will have some physical applications. It is interesting to point out that all the computations of the trigonometrical sums in this paper are based on a formula by Schwatt [12] on trigonometrical power sums, and the representation of the binomial coefficients by the residue operator. Schwatt's formula is modified slightly, only in the case of the corner-to-corner resistance, the Kirchhoff index and trigonometrical sum given by $F_1(N, l, 2)$, and $F_1(N, l, 2)$, see Section 6, this was also the case in our previous paper [1].

This paper is organized as follows: in Section 2, we give an explicit computations of the two-point resistance of the N-cycle graph and Green's function of the one-dimensional lattice, and in Section 3, we give a simple derivation of a trigonometrical sum connected with Scott's conjecture on the permanent of the Cauchy matrix. In Section 4, we consider trigonometrical sums arising in the chiral Potts model, and in the Verlinde formula of the dimension of the conformal blocks. Exact computations of the corner-to-corner and the Kirchhoff index of $2 \times N$ resistor network will be given in Section 5. In Section 6, we consider the other class of trigonometrical sums, some of which are related to number theory, and finally, in Section 7, our conclusions are given.

2. The two-point resistance of one-dimensional lattice using the residue operator

In this section, we first start with the two point resistance of the N-cycle graph computations, then, we move to the trigonometrical sum related to the two-point resistance of the one-dimensional lattice with free boundaries, that is, the path graph. The two-point resistance of the N-cycle graph between any two nodes α and β is given by the following simple closed formula [2],

$$R(l) = \frac{1}{N}\sum_{n=1}^{N-1}\frac{\sin^2(nl\pi/N)}{\sin^2(n\pi/N)} = \frac{l(N-l)}{N}, \tag{1}$$