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A dynamical formulation of one-dimensional scattering theory and its applications in optics

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HIGHLIGHTS

- Proposes a dynamical theory of scattering in one dimension.
- Derives and solves dynamical equations for scattering data.
- Gives a new inverse scattering prescription.
- Constructs optical potentials with desired scattering properties.

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ABSTRACT

We develop a dynamical formulation of one-dimensional scattering theory where the reflection and transmission amplitudes for a general, possibly complex and energy-dependent, scattering potential are given as solutions of a set of dynamical equations. By decoupling and partially integrating these equations, we reduce the scattering problem to a second order linear differential equation with universal initial conditions that is equivalent to an initial-value time-independent Schrödinger equation. We give explicit formulas for the reflection and transmission amplitudes in terms of the solution of either of these equations and employ them to outline an inverse-scattering method for constructing finite-range potentials with desirable scattering properties at any prescribed wavelength. In particular, we construct optical potentials displaying threshold lasing, antilasing, and unidirectional invisibility.

1. Introduction

Consider a possibly complex and energy-dependent scattering potential $v(x)$ that satisfies the vanishing boundary condition at infinity, i.e., $v(x) \to 0$ as $x \to \pm\infty$. The general solution of the

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time-independent Schrödinger equation,

$$-\psi''(x)+v(x)\psi=k^2\psi(x), \tag{1}$$

has the following asymptotic form.

$$\psi(x)=A_\pm e^{ikx}+B_\pm e^{-ikx} \quad \text{for } x\to\pm\infty, \tag{2}$$

where $A_\pm$ and $B_\pm$ are complex coefficients. All the scattering properties of the potential are encoded in the associated transfer matrix $\mathbf{M}$ which is defined by the condition:

$$\begin{bmatrix}A_+\\B_+\end{bmatrix}=\mathbf{M}\begin{bmatrix}A_-\\B_-\end{bmatrix}. \tag{3}$$

The entries M_{ij} of $\mathbf{M}$ are related to the left/right reflection and transmission amplitudes, $R^{r/l}$ and T, according to [1,2]

$$M_{11}=T-R^lR^r/T,\qquad M_{12}=R^r/T,\qquad M_{21}=-R^l/T,\qquad M_{22}=1/T. \tag{4}$$

The transfer matrix has a number of interesting properties. It has a unit determinant. The zeros of M_{22} in the complex k^2-plane correspond to the bound states, resonances, and anti-resonances of v. In particular, the real zeros yield the zero-width resonances that are called spectral singularities [1,3]. In optics, these give rise to threshold lasing [4]. The real zeros of M_{11} are the energy values at which the potential acts as a coherent perfect absorber (CPA) [5]. This is also known as an anti-laser [6]. The real zeros of M_{12} and M_{21} are respectively the energies at which v is reflectionless from the right and the left. If at such an energy only one of M_{12} and M_{21} vanishes while $M_{22}=1$, the system displays unidirectional invisibility [7]. The latter has important applications in devising unidirectional optical devices, and has been a subject of extensive theoretical [8–10,7] and experimental studies [11] in the past two years. The problem of constructing optical potentials with any of the above-mentioned properties is therefore of utmost importance. In this article we develop a formulation of scattering theory that, besides its conceptual and practical advantages, offers a simple solution for this kind of inverse scattering problems.

An important property of the transfer matrix is its composition property [12]: Consider the truncated potential

$$v_a(x):=v(x)\theta(a-x)=\begin{cases}v(x) & \text{for } x\le a,\\ 0 & \text{for } x>a,\end{cases} \tag{5}$$

where a is a real number and $\theta(x)$ is the step function with values 0 and 1 respectively for $x<0$ and $x\ge 0$. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be respectively the transfer matrix for v_a and $v-v_a$. Then, $\mathbf{M}_2\mathbf{M}_1=\mathbf{M}$. This relation plays a particularly useful role in modeling and the numerical investigation of various physical phenomena in optics [13], condensed matter physics [14], and acoustics [15].

The composition property of the transfer matrix is shared by another quantum mechanical quantity of central importance, namely the evolution operator $U(t,t_0)$ of any Hamiltonian operator; if we denote $U(t,t_0)$, $U(t_1,t_0)$, and $U(t,t_1)$ respectively by U, U_1, and U_2, with $t_1\in[t,t_0]$, we have $U_2U_1=U$. This simple observation leads to the natural question whether we can relate $\mathbf{M}$ to the evolution operator of a 2×2 matrix Hamiltonian. As we show below the answer to this question is in the affirmative.

2. Transfer matrix given by an evolution operator

Consider the two-component state vector:

$$\Psi:=\frac{1}{2}\begin{bmatrix}e^{-ikx}(\psi-ik^{-1}\psi')\\ e^{ikx}(\psi+ik^{-1}\psi')\end{bmatrix}. \tag{6}$$

It is easy to show that if we express Ψ as a function of $\tau:=kx$, then ψ is a solution of (1) if and only if Ψ satisfies the Schrödinger equation, $i\frac{d}{d\tau}\Psi(\tau)=\mathscr{H}(\tau)\Psi(\tau)$, for the following singular, traceless,