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Separability and dynamical symmetry of Quantum Dots

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HIGHLIGHTS

- The separability of Quantum Dots is derived from that of the perturbed Kepler problem.
- Harmonic perturbation with 2:1 anisotropy is separable in parabolic coordinates.
- The system has a conserved Runge–Lenz type quantity.

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ABSTRACT

The separability and Runge–Lenz-type dynamical symmetry of the internal dynamics of certain two-electron Quantum Dots, found by Simonović et al. (2003), are traced back to that of the perturbed Kepler problem. A large class of axially symmetric perturbing potentials which allow for separation in parabolic coordinates can easily be found. Apart from the 2:1 anisotropic harmonic trapping potential considered in Simonović and Nazmitdinov (2013), they include a constant electric field parallel to the magnetic field (Stark effect), the ring-shaped Hartmann potential, etc. The harmonic case is studied in detail.

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1. Introduction

A two-electron Quantum Dot (QD) in a perpendicular magnetic field, described by the Hamiltonian,

$$H = \sum_{a=1}^{2} \left[\frac{1}{2M} (\boldsymbol{p}_a - e\boldsymbol{A}_a)^2 + U(\boldsymbol{r}_a) \right] - \frac{a}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|}, \tag{1.1}$$

where the confining potential is that of an axially symmetric oscillator [1,2],

$$U(\boldsymbol{r}) = \frac{M}{2} \left[\omega_0^2 (x^2 + y^2) + \omega_z^2 z^2 \right], \tag{1.2}$$

may carry unexpected symmetries. Firstly, the system splits, consistently with Kohn's theorem, into center-of-mass and relative motion and the former system carries a Newton–Hooke type symmetry [3,4]. Secondly, for the particular values of the frequency ratios

$$\tau = \frac{\omega_z}{\sqrt{\omega_0^2 + \omega_L^2}} = 1,\ 2, \tag{1.3}$$

where ω_L is the Larmor frequency,[1] the *relative* motion becomes separable in suitable coordinates [1], which hints at further symmetry. This paper is devoted to the study of the latter, and to generalizing them to other axi-symmetric trapping potentials.

Our first step is to trace back the problem to those results found earlier for a particle without a magnetic field, $\boldsymbol{B} = 0$ [5,6]. Choosing the vector potential $\boldsymbol{A} = \frac{1}{2}B(-y, x, 0)$ and introducing $\boldsymbol{R} = (\boldsymbol{r}_1 + \boldsymbol{r}_2)/2$ and $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$, the system splits into center-of-mass and relative parts. Disregarding the first, we focus our attention at the relative motion. Following [1], the relative Hamiltonian becomes, after suitable re-definition,

$$H \equiv H_{rel} = -\frac{1}{2M^*}\left(\overrightarrow{\nabla}_\rho - eiA_\rho\right)^2 + \frac{M^*}{2}\left(\omega_0^2(x^2+y^2) + \omega_z^2 z^2\right) - \frac{a}{r}, \tag{1.4}$$

where $M^* = M/2$ is the reduced mass and we used units where $\hbar = 1$. Now putting

$$\boldsymbol{r} \to R(t)\,\boldsymbol{r}, \qquad R(t) = \begin{pmatrix} \cos\omega_L t & \sin\omega_L t \\ -\sin\omega_L t & \cos\omega_L t \end{pmatrix}, \qquad \omega_L = \frac{eB}{2M^*} \tag{1.5}$$

eliminates the vector potential altogether and the Schrödinger equation of relative motion, $\left[i\partial_t - H_{rel}\right]\psi = 0$, goes over into

$$\left[i\partial_t + \underbrace{\frac{\triangle}{2} + \frac{a}{r}}_{\text{Kepler}} - \underbrace{\frac{1}{2}(\omega_0^2 + \omega_L^2)\left(x^2+y^2\right) - \frac{1}{2}\omega_z^2 z^2}_{\textit{axi-symmetric oscillator}} \right] \psi = 0, \tag{1.6}$$

where we also assumed that $M^* = 1$.

The rotational trick (1.5) allowed us, hence, to convert the constant-magnetic-field problem into that of *the Kepler potential perturbed by an axially symmetric oscillator* [5,6]. In what follows, we only study the latter problem, since all results can be translated to the constant-magnetic context by applying (1.5) backwards. Note that in the original QD problem the electrons repel and thus $a \propto -e^2 < 0$; for completeness, we also consider here the attractive Kepler case $a > 0$. Our analysis bears also strong similarities with that of ions in a Paul trap [6].

[1] In the QD problem the Larmor frequency involves the reduced mass $M^* = M/2$, $\omega_L = eB/2M^*$.