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Light beams with general direction and polarization: Global description and geometric phase

R. Nityananda [a,b], S. Sridhar [c,*]

[a] TIFR Centre for Interdisciplinary Sciences, 21, Brundavan colony, Narsingi, Hyderabad 500 089, India

[b] National Centre for Radio Astrophysics, TIFR, Pune 411 007, India

[c] Raman Research Institute, Sadashivanagar, Bangalore 560 080, India

HIGHLIGHTS

- We construct a polarization basis for light which is smooth in all directions.
- Proof that the manifold of all polarizations and directions is $S^2 \times S^2$.
- Formula for the geometric phase for paths in $S^2 \times S^2$, generalizing earlier work.

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ABSTRACT

We construct the manifold describing the family of plane monochromatic light waves with all directions, polarizations, phases and intensities. A smooth description of polarization, valid over the entire sphere S^2 of directions, is given through the construction of an orthogonal basis pair of *complex* polarization vectors for each direction; any light beam is then uniquely and smoothly specified by giving its direction and two complex amplitudes. This implies that the space of all light beams is the six dimensional manifold $S^2 \times \mathbb{C}^2 \setminus \{\mathbf{0}\}$, the (untwisted) Cartesian product of a sphere and a two dimensional complex vector space minus the origin. A Hopf map (i.e. mapping the two complex amplitudes to the Stokes parameters) then leads to the four dimensional manifold $S^2 \times S^2$ which describes beams with all directions and polarization states. This product of two spheres can be viewed as an *ordered* pair of two points on a single sphere, in contrast to earlier work in which the same system was represented using Majorana's mapping of the states of a spin one quantum system to an *unordered* pair of points on a sphere.

* Corresponding author.
E-mail addresses: rajaram@ncra.tifr.res.in (R. Nityananda), ssridhar@rri.res.in (S. Sridhar).

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This is a different manifold, CP^2, two dimensional complex projective space, which does not faithfully represent the full space of all directions and polarizations. Following the now-standard framework, we exhibit the fibre bundle whose total space is the set of all light beams of non-zero intensity, and base space $S^2 \times S^2$. We give the $U(1)$ connection which determines the geometric phase as the line integral of a one-form along a closed curve in the total space. Bases are classified as globally smooth, global but singular, and local, with the last type of basis being defined only when the curve traversed by the system is given. Existing as well as new formulae for the geometric phase are presented in this overall framework.

1. Introduction and summary

Bertolotti [1] formulated the evolution of linear polarization as light traverses a space curve in an inhomogeneous but locally isotropic medium in the geometrical optics limit. He concluded that the electric vector is parallel-transported with respect to a connection derived from a conformally flat metric, where Euclidean distances are scaled by the local value of the refractive index. Rytov [2] derived this evolution law independently, by a WKB treatment of Maxwell's equations, and also expressed it as a phase difference per unit length between the two circular polarizations, proportional to the torsion of the space curve. Vladimirskii [3] brought out the following geometrical implication: polarization vectors live in the tangent plane to the sphere of directions and undergo parallel displacement as the direction changes. This implies that after the tangent vector to the curve returns to its original value (e.g. after one turn of a helix), the polarization rotates by an angle equal to the solid angle enclosed by the closed trajectory of the tangent vector on this sphere.

Pancharatnam [4], in the context of novel interference patterns shown by absorbing biaxial crystals, formulated the phase which now bears his name, equal to one-half of the solid angle traversed on the Poincare sphere which represents polarization states. The work of Berry [5] on the phase change of a quantum state, evolving adiabatically as the Hamiltonian describes a closed path in a parameter space, and its later generalizations, provides the natural framework in which to discuss this class of optical situations; see e.g. [6,7] for reviews of early work. The Berry or geometric phase depends on the path traversed in the parameter space (i.e. the sphere of directions or of polarizations) but not on the rate of traversal.

Bhandari [8–10], Hannay [11] and Tavrov et al. [12], among others, treated geometric phases under the simultaneous evolution of direction and polarization. This can be viewed as occurring in a four dimensional space, whose global structure is naturally of interest. Bhandari used part of a spin one Hilbert space to represent polarization for a given direction, and then rotation operators to represent states for different directions. Hannay used the ray space (the space of physical states) of a spin one Hilbert space to represent both polarization and directions. Such a ray space is constructed from a three dimensional Hilbert space by identifying vectors differing only in normalization and phase; it is denoted by CP^2, two dimensional complex projective space. The spin one description is indeed a faithful mapping of the elliptical orbits traversed by the electric fields of the light beams being considered, which are the same as the orbits of a three dimensional isotropic harmonic oscillator. However, two problems prevent CP^2 from being the four dimensional space that faithfully represents all directions and polarizations. One is that a given ellipse traversed by the electric vector in a plane in (real) three dimensional space can correspond to two opposite directions of propagation. This could be resolved by doubling CP^2, using one copy for each sense of circular or elliptic polarization. However, another problem arises on the boundary between these two regions: an electric field which is linearly polarized along a given direction can belong, not just to two but, to an entire circle of directions of propagation in a plane perpendicular to it; i.e. the subspace of linear polarizations in the spin one model is two dimensional, whereas it should really be three dimensional. Thus the correct global nature of this space is still not clarified in the existing work, and we address this in our paper.