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Theoretical formulation of finite-dimensional discrete phase spaces: I. Algebraic structures and uncertainty principles

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ABSTRACT

We propose a self-consistent theoretical framework for a wide class of physical systems characterized by a finite space of states which allows us, within several mathematical virtues, to construct a discrete version of the Weyl–Wigner–Moyal (WWM) formalism for finite-dimensional discrete phase spaces with toroidal topology. As a first and important application from this *ab initio* approach, we initially investigate the Robertson–Schrödinger (RS) uncertainty principle related to the discrete coordinate and momentum operators, as well as its implications for physical systems with periodic boundary conditions. The second interesting application is associated with a particular uncertainty principle inherent to the unitary operators, which is based on the Wiener–Khinchin theorem for signal processing. Furthermore, we also establish a modified discrete version for the well-known Heisenberg–Kennard–Robertson (HKR) uncertainty principle, which exhibits additional terms (or corrections) that resemble the generalized uncertainty principle (GUP) into the context of quantum gravity. The results obtained from this new algebraic approach touch on some fundamental questions inherent to quantum mechanics and certainly represent an object of future investigations in physics.

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1. Introduction

Recently, it has appeared an appreciable number of papers in literature proposing similar theoretical frameworks with intrinsic mathematical virtues for treating a wide number of physical systems which are characterized by a finite space of states [1] – in these descriptions, the state spaces are N-dimensional pre-Hilbert spaces on which an inner product is defined [2]. Besides, in connection with these state spaces, it should be stressed that quantum representations of N^2-dimensional discrete phase spaces can also be constructed from this context [3–7] and worked out in order to describe the quasiprobability distribution functions [8–16] in complete analogy with the continuous case. In what concerns the huge range of potential applications associated with the discrete distribution functions, it covers different topics of particular interest in physics, such as quantum information science [17–20], spin-tunneling effects [21,22], open quantum systems [23], and magnetic molecules [24]. In this way, relevant operators whose kinematical and/or dynamical contents carry all the necessary information for describing those physical systems are now mapped in such finite-dimensional discrete phase spaces, this fact being interpreted as a step forward in our comprehension on certain non-obvious discrete symmetries and fundamental physical properties inherent to the aforementioned operators.

The main goal of this paper is to present, via an *ab initio* construction, a self-consistent algebraic framework for finite-dimensional discrete phase spaces with toroidal topology, which embodies an important set of convenient inherent mathematical properties, within other virtues, that leads us to investigate some fundamental questions related to physical systems described by N-dimensional (discrete) state spaces for odd N. In fact, this theoretical formulation can be viewed as a discrete version of the WWM approach [25] for continuous phase spaces, with immediate applications in quantum information theory and quantum computation [26]. In addition, our results also seem to be quite suitable to deal with a wide range of problems associated with statistical mechanics (spin systems [27]) and foundations of quantum mechanics (entanglement effects in finite-dimensional multipartite physical systems [28]). The advantages and/or disadvantages of the present quantum-mechanical formulation are intrinsically connected with the inherent topology of the finite physical system under investigation, this fact being opportunely discussed in the text.

Next, let us briefly mention some important points of this particular construction process which constitute the first part of this paper. Initially formulated by Schwinger [29], the technique of constructing unitary operator bases and the associated algebraic structure represent, in this context, two essential basic elements that lead us to define a mod(N)-invariant unitary operator basis with unique characteristics: (i) it is basically constructed by means of a unitary transformation, via discrete displacement generator, on the parity operator $\mathcal{P}$; (ii) its mathematical properties allow to conclude that it is a complete orthonormal operator basis; and consequently, (iii) all the necessary quantities for describing the kinematical and dynamical contents of a given finite physical system can be mapped upon a well-established finite-dimensional discrete phase space. The discrete coherent states are then formally introduced into our quantum-algebraic framework, since they represent an important example of finite quantum states with periodic boundary conditions; moreover, it is worth stressing that certain well-known analyticity properties (namely, the non-orthogonality and completeness relations) for the continuous counterpart [30,31] were properly evaluated in this case, the Jacobi theta functions [32,33] playing a crucial rule in such a constructive process.

The second part of this paper is focused basically on two important possibilities of uncertainty principles in finite-dimensional discrete phase spaces. The first situation to be considered certainly represents an opportune moment for discussing the RS uncertainty principle [34] associated with the discrete coordinate and momentum operators. To develop such a particular task, it is necessary to adapt the previous formalism, initially conceived for unitary operators, in order to include within its scope the Hermitian coordinate and momentum operators. Then, after some basic arrangements, such an algebraic apparatus can be used for evaluating, among other things, moments and mean values of commutation and anticommutation relations involving these operators for any finite physical system where periodic boundary conditions do not apply. It is worth emphasizing that the RS uncertainty principle and its inherent implications open in this context an important window of future searches on finite quantum states [35]. The second situation concerns a particular realization of the unitary operators: by means of complex exponentials in which the arguments are written in terms of the