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Quantum dissipation from power-law memory

Vasily E. Tarasov *

Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow 119991, Russia

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ABSTRACT

A new quantum dissipation model based on memory mechanism is suggested. Dynamics of open and closed quantum systems with power-law memory is considered. The processes with power-law memory are described by using integration and differentiation of non-integer orders, by methods of fractional calculus. An example of quantum oscillator with linear friction and power-law memory is considered.

1. Introduction

A new quantum dissipation model based on memory mechanism is suggested. Dynamics of open and closed quantum systems with power-law memory is considered. An example of quantum oscillator with linear friction and power-law memory is considered. The processes with power-law memory are described by using integration and differentiation of non-integer orders, by methods of fractional calculus [1,2]. Fractional calculus is a theory of integrals and derivatives of any arbitrary real (or complex) order. It has a long history from 30 September 1695, when the derivatives of order $\alpha = 1/2$ has been described by Leibniz in a letter to L'Hospital [3,4]. The fractional differentiation and fractional integration go back to many great mathematicians such as Leibniz, Liouville, Riemann, Abel, Riesz, Weyl. There are the special journals: "Fractional Calculus and Applied Analysis"; "Fractional Differential Calculus"; "Communications in Fractional Calculus". The first book dedicated specifically to the theory of fractional integrals and derivatives, is the book by Oldham and Spanier [5] published in 1974. There exists the remarkably comprehensive encyclopedic-type monograph by Samko et al. [1], which was published in Russian in 1987 and in English in 1993. In 2006 Kilbas et al. published a very important and remarkable book [2], where one can find a modern encyclopedic, detailed and rigorous theory of fractional differential equations. Applications of fractional calculus in physics are described

* Tel.: +7 499 939 5989; fax: +7 499 939 0397.
E-mail address: tarasov@theory.sinp.msu.ru.

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in the books [6–14]. In general, many usual properties of the ordinary (first-order) derivative D_t are not realized for fractional derivative operators D_t^α. For example, a product rule, chain rule, semigroup property have strongly complicated analogs for the operators D_t^α.

The fractional calculus is a powerful tool to describe physical systems that have long-time memory. Fractional differentiation with respect to time is characterized by long-term memory effects that correspond to intrinsic dissipative processes in the physical systems [15–18]. The memory effects to discrete maps mean that their present state evolution depends on all past states. Note that a power-law memory has been detected for fluctuation within a single protein molecule [19]. The nonholonomic systems with generalized constraints to describe a long-time memory are considered [20]. The electrodynamics of dielectric media is described as a fractional temporal electrodynamics [21–23]. The discrete maps with memory are obtained from the fractional differential equations of classical dynamical systems [24–27].

2. Derivatives and integrals of non-integer order

There are many different definitions of fractional integrals and derivatives of non-integer orders [1,2].

2.1. A generalization of Cauchy's differentiation formula

Let G be an open subset of the complex plane $\mathbb{C}$, and $f : G \to \mathbb{C}$ is a holomorphic function:

$$f^{(n)}(x) = \frac{n!}{2\pi i} \oint_L \frac{f(z)}{(z-x)^{n+1}}\, dz. \tag{1}$$

A generalization of (1) has been suggested by Sonin and Letnikov in 1872 in the form

$$D_x^\alpha f(x) = \frac{\Gamma(\alpha+1)}{2\pi i} \oint_L \frac{f(z)}{(z-x)^{\alpha+1}}\, dz, \tag{2}$$

where $\alpha \in \mathbb{R}$ and $\alpha \neq -1, -2, -3, \dots$ (see Theorem 22.1 in the book by Samko et al. [1]). Expression (2) is also called Nishimoto derivative.

2.2. A generalization of finite difference

The differentiation of integer order n can be defined by

$$D_x^n f(x) = \lim_{h\to 0} \frac{\Delta_h^n f(x)}{h^n}, \tag{3}$$

where Δ_h^n is a finite difference of integer order n:

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x-kh). \tag{4}$$

The difference of a fractional order $\alpha > 0$ is defined by the infinite series

$$\Delta_h^\alpha f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x-kh), \tag{5}$$

where the binomial coefficients are

$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)}. \tag{6}$$

The left-and right-sided Grünwald–Letnikov derivatives of order $\alpha > 0$ are defined by

$$^{GL}D_{x\pm}^\alpha f(x) = \lim_{h\to 0} \frac{\nabla_{\mp h}^\alpha f(x)}{h^\alpha}. \tag{7}$$