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Some results on the dynamics and transition probabilities for non self-adjoint hamiltonians

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ABSTRACT

We discuss systematically several possible inequivalent ways to describe the dynamics and the transition probabilities of a quantum system when its hamiltonian is not self-adjoint. In order to simplify the treatment, we mainly restrict our analysis to finite dimensional Hilbert spaces. In particular, we propose some experiments which could discriminate between the various possibilities considered in the paper. An example taken from the literature is discussed in detail.

1. Introduction

In ordinary quantum mechanics one of the fundamental axiom of the whole theory is that the hamiltonian H of the physical system must be self-adjoint: $H = H^\dagger$. This condition, shared also by all the *observables* of the system, is important since it ensures that the eigenvalues of these observables, and of the hamiltonian in particular, are real quantities. However, this is not a necessary condition, and in fact several physically motivated examples exist in the literature concerning non self-adjoint operators whose spectra consist of only real eigenvalues.

However, $H = H^\dagger$ has an extra bonus, since the time evolution deduced out of H is unitary and, being so, preserves the total probability: if $\Psi(t)$ is a solution of the Schrödinger equation $i\dot{\Psi}(t) = H\Psi(t)$, then $\|\Psi(t)\|^2$ does not depend on time. This is clear since $\Psi(t) = e^{-iHt}\Psi(0)$, and since

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$U_t = e^{-iHt}$ is unitary. Of course, this is false if $H \neq H^\dagger$, and in fact, in this case, $\|\Psi(t)\|^2$ does indeed depend on time, in general. Sometimes this is exactly what one looks for: in many simple systems in quantum optics, for instance, non self-adjoint hamiltonians are used to describe some decay, so that there is no reason for the probability to be preserved in time. Other times, one would prefer to avoid any damping, so that the aim is to find some way to *recover unitarity* even when $H \neq H^\dagger$. This is particularly interesting for people in the PT-community, who quite often work with hamiltonian operators which are not self-adjoint, but simply pseudo-symmetric or PT-symmetric, [1,2], and in fact several attempts have been proposed along the years by different authors to discuss this and other aspects of time evolution for systems driven by non self-adjoint hamiltonians. Here we refer to [3–10], and references therein. However, in our opinion, much more can be said, and using a rather general approach. This is exactly what we will do here, in the next section, considering the cases in which the eigenvalues of H are all real and commenting on the situation in which some eigenvalues are complex.

In all this paper we will work with finite-dimensional Hilbert spaces. This has two nice consequences: the first one is that all the operators involved are bounded (hence, everywhere defined) and the inverse, when it exists, is bounded as well. In fact, we are dealing with matrices. The second consequence is that we can easily, quite often, discuss examples in terms of pseudo-fermions (PFs), [11,12], as we have already recently shown in [13]. We should stress that, contrarily to what often stated in the literature, going from a finite to an infinite dimensional Hilbert space is an absolutely non trivial task. Therefore, most of our claims, though giving indications also in this latter case, are rigorously true only in the present, finite-dimensional, settings. We will comment more on this aspect all along the paper.

This article is organized as follows:

In the next section we discuss the general functional structure associated to a non self-adjoint hamiltonian, and its dynamics. We also comment briefly on the case of non purely real eigenvalues and on finite temperature equilibrium states. In Section 3 we propose different definitions of transition probability functions, and we discuss a possible strategy to discriminate between them. This is, in fact, the core of our paper since it could be used, in principle, to deduce which are the *correct* Hilbert space, scalar product, norm and adjoint, or, more explicitly, which definitions reproduce the experimental data. This proposal is made more precise in Section 4, with the aid of an explicit example, originally introduced in [4] and discussed here adopting a simple and general pseudo-fermionic representation. Section 5 contains our conclusions. To keep the paper self-consistent, we list some definitions and results on PFs in the Appendix.

2. A general settings for $H \neq H^\dagger$

As we have already said, in this paper we will focus on the easiest situation, i.e. on finite dimensional Hilbert spaces. In this way our operators are finite matrices. The main ingredient is an operator (i.e. a matrix) H, acting on the vector space $\mathbb{C}^{N+1}$, with $H \neq H^\dagger$ and with exactly $N+1$ distinct eigenvalues E_n, $n = 0, 1, 2, \ldots, N$. Here, the adjoint $H^\dagger$ of H is the usual one, i.e. the complex conjugate of the transpose of the matrix H. Because of what follows, and in order to fix the ideas, it is useful to remind here that the adjoint of an operator X, $X^\dagger$, is defined in terms of the *natural* scalar product $\langle .,. \rangle$ of the Hilbert space $\mathcal{H} = \left(\mathbb{C}^{N+1}, \langle .,. \rangle\right)$: $\langle Xf, g\rangle = \left\langle f, X^\dagger g\right\rangle$, for all $f, g \in \mathbb{C}^{N+1}$, where $\langle f, g\rangle = \sum_{k=0}^{N} \overline{f_k}\, g_k$, with obvious notation. We will consider separately the case in which all the eigenvalues E_n are real and the situation in which some are complex. In both cases we will assume that each E_n has multiplicity one.

Before starting, it is necessary to clarify some notation adopted in this paper: we will use $\mathbb{C}^{N+1}$ any time we want to stress the nature of vector space of our vectors. When it is important to stress the topological (i.e. the scalar products and the norms) aspects of this set, we will use $\mathcal{H}$ instead of $\mathbb{C}^{N+1}$ (and, later, $\mathcal{H}_\varphi$ or $\mathcal{H}_\Psi$). Before starting with our analysis, it is surely worth stressing that, with a different language, some of the results discussed in Section 2 can be found in the literature, see [2,8–10,14] for instance. We have decided to include these statements here for several reasons: first, they are useful to fix our notation. Secondly, some of the proofs discussed here are different, or cannot be found, in the existing literature. Last but not least, we want to keep an eye to possible extensions of