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Infinitely many singular interactions on noncompact manifolds

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HIGHLIGHTS

- Schrödinger-operator for infinitely many singular interactions on noncompact manifolds.
- Proof of the finiteness of the ground-state energy.

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ABSTRACT

We show that the ground state energy is bounded from below when there are infinitely many attractive delta function potentials placed in arbitrary locations, while all being separated at least by a minimum distance, on two dimensional non-compact manifold. To facilitate the reading of the paper, we first present the arguments in the setting of Cartan–Hadamard manifolds and then subsequently discuss the general case. For this purpose, we employ the heat kernel techniques as well as some comparison theorems of Riemannian geometry, thus generalizing the arguments in the flat case following the approach presented in Albeverio et al. (2004).

1. Introduction

In this article, we would like to revisit the study of point interactions in Riemannian manifolds, as presented in Ref. [1] to understand the case of infinitely many point centers on a non-compact manifold. Since the problem by its nature is nonperturbative and is defined on a Riemannian manifold, the authors of Ref. [1] study the model through the resolvent of the full Hamiltonian. The resolvent

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contains an additional part, expressed via the inverse of an operator, so-called principal operator, formulated completely with the help of the heat kernel of the Laplace operator on the manifold (this form of the resolvent is known as Krein's formula in the mathematics literature). The use of heat kernel of the Laplacian within the resolvent has a definite advantage for addressing both the geometry and the renormalization related issues on the same footing. This becomes especially transparent when one deals with singular interactions supported on curves and submanifolds, which received some attention recently for curves [2–10] and surfaces [11–15] embedded in flat space. When the ambient space becomes a curved manifold, using heat kernel techniques turns out to be essential with which regularization is naturally accomplished, as it is developed in Refs. [16–18]. Thus, it is, in these references, shown that this method upon which the whole formalism is based, allows one to work in a natural environment to analyze the interrelations between the geometry of the manifold, the bound-state energy, and the renormalization aspects of these models.

In this article our pursuit is to extend the study of countably infinitely many point interactions in $\mathbb{R}^2$, as it is presented in Chapter 3 in the authoritative book [19], to the case in which the underlying space is replaced by a general two dimensional noncompact manifold. An equally interesting and difficult problem appears in the one dimensional case when the delta functions are all arbitrarily distributed with no minimum distance conditions and with arbitrary strengths. The spectral aspects of this problem and various conditions on the strengths and distances are thoroughly investigated in the recent work by Kostenko and Malamud [20]. It is an interesting challenge to understand the spectral aspects of the present problem under more general assumptions for the coupling strengths. A more recent study on the spectral aspects of infinitely many point interactions in three dimensional flat space is studied in Ref. [21], where the reader can find further references on the subject. Our main result can be summarized as follows: Consider two dimensional noncompact manifold whose sectional curvature is bounded from below. On such a manifold, we introduce an infinite number of point interactions on arbitrary locations. If all the bound state energies supported by these point centers are bounded from below by a common value μ_* and moreover all the distances between these points are bounded from below by a minimum distance $d_{\min}$, then one can compute a lower bound to the ground state energy, expressed in terms of $d_{\min}$, μ_* and the geometric data of the manifold. This problem has importance especially when one thinks of these delta function centers as models of impurities, appearing in random locations inside the manifold, and avoiding one another by a certain distance by considerations of energy. It is certainly a very interesting and a very challenging problem, which we plan to return in a subsequent work, to consider singular interactions supported both on infinitely many curves or submanifolds, all of which are embedded in some non-compact Riemannian manifolds, extending what is presented in Refs. [16–18]. Some results in this direction are obtained for special geometries in flat space in Ref. [22].

The plan of the article is as follows: in Section 2, the model will be constructed as a limit of well-defined projection operators through the aforementioned resolvent formula. Afterwards our choice of regularization of the principal operator will be shown to suffice to renormalize the model. In Section 3, the proof of which the ground-state energy is finite will be presented by exploiting some remarkable estimates of the heat kernel of Cartan–Hadamard manifolds, using the Holmgren bound for the norm of countably infinite matrices, and lastly relying on the remarkable comparison theorem of Toponogov. In Section 4, we present the proof for general noncompact manifolds, and emphasize the main changes. It is implicit that the construction of the resolvent is also accomplished following the approach presented in [19], since this does not require any new ideas, we only present the details of lower bound for the ground state energy.

2. Construction of the renormalized Hamiltonian

In this section, we review the construction of the resolvent operator on two dimensional manifolds corresponding to the Hamiltonian with countably infinitely many singular interactions, that are at least separated from each other by a global minimum distance.

Let us consider a Hamiltonian operator H in $\mathcal{L}^2(\mathcal{M})$ with countably infinitely many singular interactions. A convenient way to express the interaction part of the Hamiltonian in a manifold is