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New implicitly solvable potential produced by second order shape invariance

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HIGHLIGHTS

- New potential with 2nd order irreducible shape invariance was constructed.
- The connection conditions at the singularity of potential were obtained.
- The explicit expressions for all wave functions were derived.
- The implicit equation for the energy spectrum was obtained.

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ABSTRACT

The procedure proposed recently by Bougie et al. (2010) to study the general form of shape invariant potentials in one-dimensional Supersymmetric Quantum Mechanics (SUSY QM) is generalized to the case of Higher Order SUSY QM with supercharges of second order in momentum. A new shape invariant potential is constructed by this method. It is singular at the origin, it grows at infinity, and its spectrum depends on the choice of connection conditions in the singular point. The corresponding Schrödinger equation is solved explicitly: the wave functions are constructed analytically, and the energy spectrum is defined implicitly via the transcendental equation which involves Confluent Hypergeometric functions.

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1. Introduction

Starting from the paper [1], the method of Supersymmetric Quantum Mechanics (SUSY QM) became a new effective tool [2,3] for investigation of different problems in conventional Quantum Mechanics. From a mathematical point of view, this method – for one-dimensional space – is a reformulation of so called Darboux transformation of Sturm–Liouville equation well known to mathematicians since more than hundred years [4]. SUSY QM approach was helpful for investigation of different problems in Quantum Mechanics and a lot of generalizations were proposed. In particular, the multi-dimensional generalization [5,6,3] of SUSY QM which led to many new results, mostly for two-dimensional case [7,6] has to be mentioned. This multi-dimensional approach can be considered also as a generalization of old Darboux transformations.

Among new ideas provided by SUSY QM method, one – the shape invariance – is of special significance [8–11]. It allows to solve some quantum models exactly or quasi-exactly. The list of known exactly solvable one-dimensional models is very restricted (approximately, about ten potentials are in this list). Supersymmetric transformations provide the partnership between pairs of potentials and therefore lead to new solvable potentials [12], though sometimes of rather complicated analytical form. The method of shape invariance gives a very elegant *algebraic* algorithm to construct exactly solvable models without direct solution of differential equations. All previously known [13] one-dimensional exactly solvable systems were shown to be shape invariant ones [14]. Also, this approach provides analytical solution of several nontrivial two-dimensional problems [15,16].

Recently, the general investigation of additive form of shape invariance for superpotentials without explicit dependence on parameters was performed by J. Bougie et al. [9,10]. It was demonstrated that no additional shape invariant models in one-dimensional case can be constructed. This result was obtained in the framework of standard SUSY QM with supercharges of first order in derivatives. Meanwhile, it is known [17–19] that such standard SUSY QM does not exhaust all opportunities to fulfill the generalized SUSY algebra for Superhamiltonian $\hat{H}$ and supercharges $\hat{Q}^{\pm}$. Higher order supercharges are also possible, and for example supercharges of second order in momentum lead to some new results [19].

In the present paper just a potential which fulfills the second order supersymmetry will be produced by the condition of shape invariance of second order. This potential is of very compact form, it depends on arbitrary parameter but has a strong singularity of g/x^2 kind with $g \in (-1/4, 0)$. This last property forces to construct a suitable self-adjoint extension of the Hamiltonian H, i.e. the suitable class of functions where H acts. The paper is organized as follows. Section 2 contains a brief summary of one-dimensional SUSY QM and shape invariance. A shape invariant potential is built in Section 3 by means of second order supercharges. The direct analytical solution of the Schrödinger equation is given in Section 4. The problem of a suitable class of functions belonging to the domain of the self-adjoint extension of H is discussed in Section 5 in terms of connection conditions at the origin. Section 6 includes derivation of the Spectrum Generating equation and its analysis, and finally, some conclusions are given in Section 7.

2. SUSY QM and shape invariance

The main ingredients of one-dimensional SUSY Quantum Mechanics and of shape invariance will be briefly presented in this section (see details in Refs. [2]). An arbitrary one-dimensional Hermitian Schrödinger Hamiltonian with potential $V(x)$ and mass $m = 1/2$ can be factorized:

$$H = Q^+Q^- = -\hbar^2\,\partial^2 + V(x); \quad \partial \equiv \frac{d}{dx} \tag{1}$$

by means of the first order differential operators $Q^{\pm}$; $Q^+ = (Q^-)^{\dagger}$. Directly by construction, the superpartner Hamiltonian

$$\tilde{H} = Q^-Q^+ = -\hbar^2\,\partial^2 + \tilde{V}(x) \tag{2}$$