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Exact sum rules for inhomogeneous systems containing a zero mode

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HIGHLIGHTS

- We discuss the sum rules of the eigenvalues of inhomogeneous systems containing a zero mode.
- We derive the explicit expressions for sum rules of order one and two.
- We perform accurate numerical tests of these results for three examples.

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ABSTRACT

We show that the formulas for the sum rules for the eigenvalues of inhomogeneous systems that we have obtained in two recent papers are incomplete when the system contains a zero mode. We prove that there are finite contributions of the zero mode to the sum rules and we explicitly calculate the expressions for the sum rules of order one and two. The previous results for systems that do not contain a zero mode are unaffected.

1. Introduction

This paper aims to derive exact expressions for the sum rules involving the eigenvalues of inhomogeneous systems described by the Helmholtz equation over a finite region Ω in d dimensions

$$(-\Delta)\Psi_n(x_1,\ldots,x_d) = E_n\Sigma(x_1,\ldots,x_d)\Psi_n(x_1,\ldots,x_d) \tag{1}$$

where $\Sigma(x_1,\ldots,x_d) > 0$ for $(x_1,\ldots,x_d) \in \Omega$ and the eigenfunctions $\Psi_n(x_1,\ldots,x_d)$ obey boundary conditions on $\partial\Omega$ which allow for the presence of a vanishing eigenvalue ("zero-mode").

The possibility of obtaining *exact* results is certainly appealing since these may provide useful information on a physical system, even when the eigenvalues and the eigenfunctions of the problem cannot be obtained exactly or when perturbation theory is not applicable.

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Spectral sum rules have been obtained in the past for a number of problems, which include, for example, the sum rules for quantum mechanical anharmonic oscillators [1,2], for potentials $V(r) = gr^p$ $(p > 0, g > 0)$ [3], for Aharonov–Bohm quantum billiards [4,5], for the zeros of Bessel functions [3,5,6], for the Selberg's zeta function for compact Riemann surfaces [7] (see also [8]), for quantum mechanical one dimensional potentials [9], for two dimensional domains close to the unit disk [10], for the cardioid and related domains [11], for certain $\mathcal{PT}$-symmetric hamiltonians [12,13].

In the case of Ref. [12] the spectral zeta function $Z(1) = \sum_n 1/E_n$ for the $\mathcal{PT}$-symmetric hamiltonian $\hat{H} = \hat{p}^2 + ix^3$ was calculated exactly and used to conclude that the spectrum is entirely real by comparing the exact sum rule with the approximate sum rule obtained estimating the lower eigenvalues numerically and the higher end of the spectrum using the WKB approximation.

More recently, the sum rules associated to the eigenvalues of Eq. (1) have been studied by myself in two papers, Refs. [14,15], deriving explicit expressions for these sum rules in d dimensions and for different boundary conditions obeyed at the borders. The case of $\mathcal{PT}$-symmetric strings was also investigated in Ref. [16] for Dirichlet boundary conditions.

Although the analysis carried out in those papers was supposed to apply to the cases in which a zero mode is either absent or present in the spectrum, we have found out that the latter case requires a separate treatment, because of the additional contributions stemming from the zero mode which were not taken into account in Refs. [14,15]. In this paper we derive these contributions and provide explicit expressions for the sum rules of orders one and two.

The paper is organized as follows: in Section 2 we extend the results of Refs. [14,15], obtaining exact expressions for the sum rules of order one and two; in Section 3 we apply these formulas to problems in one and two dimensions and compare the exact results with highly precise numerical results; in Section 4 we review the main results of this paper and draw our conclusions. The details of the calculation of perturbative expression for the eigenvalue and eigenfunction of the zero mode are contained in the Appendix.

2. Exact sum rules for spectra containing a zero mode

As we have discussed in our previous papers, Refs. [14,15], Eq. (1) is isospectral to the equation

$$\left[\frac{1}{\sqrt{\Sigma(x_1,\ldots,x_d)}}(-\Delta)\frac{1}{\sqrt{\Sigma(x_1,\ldots,x_d)}}\right]\Phi_n(x_1,\ldots,x_d) = E_n\Phi_n(x_1,\ldots,x_d), \tag{2}$$

while their eigenfunctions are simply related by $\Psi_n(x_1,\ldots,x_d) = \Phi_n(x_1,\ldots,x_d)/\sqrt{\Sigma(x_1,\ldots,x_d)}$.

The spectrum of Eqs. (1) and (2) is bounded from below, in some cases being composed by strictly positive eigenvalues while in other cases containing also a zero mode. For example, in the case of an inhomogeneous string, discussed in Ref. [14], a zero mode appears when either Neumann or periodic boundary conditions are enforced.

We briefly describe the procedure that we have devised in our previous work to evaluate the sum rules $Z_p = \sum_n 1/E_n^p$, with $p = p_0, p_0+1, \ldots$ and p_0 being the smallest integer for which the series is convergent (in one dimension $p_0 = 1$).

We first define the operator

$$\hat{O} \equiv \frac{1}{\sqrt{\Sigma(x_1,\ldots,x_d)}}(-\Delta)\frac{1}{\sqrt{\Sigma(x_1,\ldots,x_d)}} \tag{3}$$

appearing in Eq. (2).

The inverse operator may be formally expressed in terms of the Green's function of the negative Laplacian obeying the same boundary conditions

$$\hat{O}^{-1}f = \sqrt{\Sigma(x_1,\ldots,x_d)}\int_\Omega G(x_1,\ldots,x_d,y_1,\ldots,y_d)\sqrt{\Sigma(y_1,\ldots,y_d)}f(y_1,\ldots,y_d). \tag{4}$$

Clearly the eigenvalues of this operator are just the reciprocals of the eigenvalues of Eq. (2) (for the moment being we are assuming that the zero mode is not present).