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The Kummer tensor density in electrodynamics and in gravity

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ABSTRACT

Guided by results in the premetric electrodynamics of local and linear media, we introduce on 4-dimensional spacetime the new abstract notion of a Kummer tensor density of rank four, $\mathcal{K}^{ijkl}$. This tensor density is, by definition, a cubic algebraic functional of a tensor density of rank four $\mathcal{T}^{ijkl}$, which is antisymmetric in its first two and its last two indices: $\mathcal{T}^{ijkl} = -\mathcal{T}^{jikl} = -\mathcal{T}^{ijlk}$. Thus, $\mathcal{K} \sim \mathcal{T}^3$, see Eq. (46). (i) If $\mathcal{T}$ is identified with the electromagnetic response tensor of local and linear media, the Kummer tensor density encompasses the generalized *Fresnel wave surfaces* for propagating light. In the reversible case, the wave surfaces turn out to be *Kummer surfaces* as defined in algebraic geometry (Bateman 1910). (ii) If $\mathcal{T}$ is identified with the *curvature* tensor R^{ijkl} of a Riemann–Cartan spacetime, then $\mathcal{K} \sim R^3$ and, in the special case of general relativity, $\mathcal{K}$ reduces to the Kummer tensor of Zund (1969). This $\mathcal{K}$ is related to the *principal null directions* of the curvature. We discuss the properties of the general Kummer tensor density. In particular, we decompose $\mathcal{K}$ irreducibly under the 4-dimensional linear group $GL(4, R)$ and, subsequently, under the Lorentz group $SO(1, 3)$.

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1. Introduction

1.1. Fresnel surface

We consider electromagnetic waves propagating in a homogeneous, transparent, dispersionless, and nonconducting crystal. The response of the crystal to electric and magnetic perturbations is assumed to be *local* and *linear*. The permittivity tensor ε^{ab} $(a, b = 1, 2, 3)$ of the crystal[1] is anisotropic in general, and the same is true for its *im*permeability tensor μ_{ab}^{-1}. Such materials are called special bi-anisotropic (see [4–6]). If ε^{ab} and μ_{ab}^{-1} are assumed to be symmetric, these bi-anisotropic materials are characterized by 12 independent parameters. In many applications, however, μ_{ab}^{-1} can be considered approximately to be isotropic $\mu_{ab}^{-1} = \mu_0^{-1} g_{ab}$; here g_{ab} is the 3-dimensional Euclidean metric tensor. Such cases were already studied experimentally and theoretically in the early 19th century, before Maxwell recognized the electromagnetic nature of light in 1862 (see [7]). To carry out these investigations on geometric optics, one used the notions of light ray and of wave vector, and one was aware (Young, 1801)[2] that light was transverse and equipped with a polarization vector.

At each point inside a crystal, we have a ray vector and a wave covector (one-form). It is then possible to determine the *ray surface* and its dual, the *wave surface,* for visualizing how a pulse of light is propagating. The ray surface was first constructed by Fresnel (1822) and is conventionally called *Fresnel surface,* an expression also used for the wave surface, see the popular introduction by Knörrer [8]. Since the symmetric permittivity tensor can be diagonalized, the Fresnel surface is described by 3 principal values. In the case when all of them are equal, the Fresnel surface is an ordinary 2-dimensional sphere. For two unequal parameters, the surface is the union of two shells, a sphere and an ellipsoid. These two shells touch at two points. In Fig. 1, we display such a surface for a crystal with *three different principal* values of the permittivity: $(\varepsilon^{ab}) = \begin{pmatrix} \varepsilon^1 & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & \varepsilon^3 \end{pmatrix}$ and $(\mu_{ab}^{-1}) = \mu_0^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. It is a union of two shells that meet at 4 singular points. As already recognized by Hadamard [9], the wave surfaces are the characteristics of the corresponding partial differential equations describing the wave propagation.

The classical 3-parameter Fresnel surface is naturally generalized when the anisotropy of the impermeability μ_{ab}^{-1} is taken into account. Removing also the diagonalizability requirement, one deals with *generalized Fresnel surfaces* of less than 12 parameters. Such surfaces were derived in terms of positive definite symmetric dyadics by Lindell [11], see also [12].

Moreover, it seems to be rather natural to extend ε and μ to asymmetric tensors, which emerge if dissipative processes are involved. Recently, one of us [13] derived a tensorial expression of such a generalized 18-parameter Fresnel surface. The derivation does not require the corresponding matrices to be real, symmetric, positive definite, or even invertible.

We here will derive such tensorial expressions for generalized Fresnel surfaces by proceeding differently. We include first magnetoelectric effects and subsequently look for the corresponding 4-dimensional relativistic covariant generalizations of the 3-dimensional permittivity and impermeability tensors.

1.2. Magnetoelectricity

In the 1960s, substances were found that, if exposed to a magnetic field B^a, were electrically polarized $D^a = \alpha^a{}_b B^b$ and, reciprocally, if exposed to an electric field E_a, were magnetized, $H_a = \beta_a{}^b E_b$, see O'Dell [14]. These are small effects of the order $10^{-3}\sqrt{\varepsilon_0/\mu_0}$, or smaller.

Such materials are characterized by the constitutive moduli ε^{ab}, μ_{ab}^{-1}, $\alpha^a{}_b$, $\beta_b{}^a$, which can be accommodated in a 6×6 matrix. Originally, all these moduli were assumed to obey the *symmetry*

1 The position of the indices are chosen always in accordance with the conventions of premetric electrodynamics, see [1–3].

2 Thomas Young (1773–1829), English mathematician and physicist.