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Braiding fluxes in Pauli Hamiltonian

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ABSTRACT

Aharonov and Casher showed that Pauli Hamiltonians in two dimensions have gapless zero modes. We study the adiabatic evolution of these modes under the slow motion of N fluxons with fluxes $\Phi_a \in \mathbb{R}$. The positions, $\mathbf{r}_a \in \mathbb{R}^2$, of the fluxons are viewed as controls. We are interested in the holonomies associated with closed paths in the space of controls. The holonomies can sometimes be abelian, but in general are not. They can sometimes be topological, but in general are not. We analyse some of the special cases and some of the general ones. Our most interesting results concern the cases where holonomy turns out to be topological which is the case when all the fluxons are subcritical, $\Phi_a < 1$, and the number of zero modes is $D = N - 1$. If $N \geq 3$ it is also non-abelian. In the special case that the fluxons carry identical fluxes the resulting anyons satisfy the Burau representations of the braid group.

1. Introduction

The Pauli Hamiltonian describes a non-relativistic electron with gyromagnetic constant $g = 2$

$$\mathbf{H}_p(A) = \frac{1}{2m}\left(-i\nabla - e\mathbf{A}\right)^2 \otimes \mathbb{1} - \frac{ge}{4m}\mathbf{B}\cdot\sigma - eA_0 \otimes \mathbb{1} \tag{1.1}$$

σ is the vector of Pauli matrices and $\mathbf{H}_p$ acts on spinors. We use units where $\hbar = c = 1$. The electric and magnetic fields are determined by the 4-potential $A = (A_0, \mathbf{A})$:

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A} + \nabla A_0. \tag{1.2}$$

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In 1979 Aharonov and Casher [1] observed that the Pauli operator for static magnetic field in two dimensions, $A = (0, A_x, A_y, 0)$, so $\mathbf{B} = B\,\hat{\mathbf{z}}$, has (normalizable) zero energy modes.[1] They are gapless ground states and their number D, is determined by the total magnetic flux Φ_T measured in units of quantum flux,

$$D = \lceil |\Phi_T| \rceil - 1, \quad \Phi_T = \frac{e}{2\pi} \int B\, dx \wedge dy \tag{1.3}$$

where $\lceil x \rceil$ stands for the Ceiling of x, i.e. the smallest integer $\geq x$.

We consider a magnetic field B localized on a finite number of disjoint fluxons labelled by $a = 1, \ldots, N$. The magnetic flux of the ath fluxon, Φ_a, is localized in a region of radius R_a centred at $\mathbf{r}_a$. We *do not* assume that Φ_a is quantized or that all the fluxes Φ_a are identical. We shall assume w.l.o.g. that $\Phi_T > 0$. We say that the ath fluxon is super-critical if $\Phi_a > 1$, subcritical if $\Phi_a < 1$ and critical if $\Phi_a = 1$. The fluxons are viewed as classical parameters and *not* as dynamical degrees of freedom: They *do not* have a wave function or an equation of motion.[2] (The dynamical degree of freedom is the electron wave function.)

When the ath fluxon is super-critical it can create $D_a = \lceil \Phi_a \rceil - 1 \neq 0$ zero modes which are *confined* to it, in the sense that their wave function decays (as a power law) over a typical distance $O(R_a)$ determined by the fluxon radius R_a. More interesting are the zero modes which are bound jointly by a number of separate fluxons. We shall call these solutions *free zero modes*. These states' wave functions live in between the fluxons and have typical size determined by the inter-distance $|\mathbf{r}_a - \mathbf{r}_b|$. When *all* the fluxons are subcritical, $D_a = 0$ *all* the zero modes are free: the probability of finding the charge on any of the fluxons is close to zero (as $R_a \to 0$). In general, confined and free modes coexist. The confined modes behave like the charge–flux composites one encounters in the fractional quantum Hall effect [3–6], except that here the charge is quantized but the flux is not whereas in the Hall effect it is the flux that is quantized and the charge is not. The free modes are a different kettle of fish as the composite involves a single electron jointly bound by several fluxons. As we shall see, these modes can sometimes turn the fluxons into non-abelian anyons [7,4,8]. These new "topological" objects are quite interesting.

The distinction between confined and free zero modes is meaningful when the radius of the individual fluxons, R_a, is the smallest length scale in the problem, $R_a \ll |\mathbf{r}_a - \mathbf{r}_b|$ and is sharp for point-like fluxons. The total number of free modes D_f is, as we shall see,

$$0 \leq D_f = Max \left\{ 0, \left\lceil \sum_a \Phi'_a \right\rceil - 1 \right\} \leq N - 1, \quad \Phi'_a = \Phi_a - D_a. \tag{1.4}$$

We say that the number of free modes is maximal if $D_f = N - 1$. This turns out to be the case where the fluxons become non-abelian anyons. If all the fluxons are identical then to have maximal number of free modes leading to interesting representation of the braid group requires

$$1 - \frac{1}{N} < \Phi_a < 1. \tag{1.5}$$

(The case $\Phi_a \equiv 1$ leads to a trivial representation of the pure braid group.)

To study the holonomy of the zero modes we treat the fluxon coordinates, $\mathbf{r}_a$ as (classical) adiabatic controls. The adiabatic theory we shall need and describe is of interest in its own right, since the weak electric fields generated by the slow motion of the fluxons are important for the adiabatic transport and since the zero modes are gapless (see Section 3.2 for more details). Adiabaticity means that the characteristic time scale of the controls is large compared with the characteristic time scale of the system. We shall argue that the characteristic time scale in the case of point-like fluxons is set by their

1 When $g > 2$ the zero modes turn into gapped bound states.

2 In contrast with, [2], where the fluxons have a wave-function and an equation of motion, and are assumed to carry half a unit of quantum flux.