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Perturbations of unitary representations of finite dimensional Lie groups ☆

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ABSTRACT

In quantum physical theories, interactions in a system of particles are commonly understood as perturbations to certain observables, including the Hamiltonian, of the corresponding interaction-free system. The manner in which observables undergo perturbations is subject to constraints imposed by the overall symmetries that the interacting system is expected to obey. Primary among these are the spacetime symmetries encoded by the unitary representations of the Galilei group and Poincaré group for the non-relativistic and relativistic systems, respectively. In this light, interactions can be more generally viewed as perturbations to unitary representations of connected Lie groups, including the non-compact groups of spacetime symmetry transformations. In this paper, we present a simple systematic procedure for introducing perturbations to (infinite dimensional) unitary representations of finite dimensional connected Lie groups. We discuss applications to relativistic and non-relativistic particle systems.

1. Introduction

In quantum theories of interacting particles, it is common to treat relevant physical observables as the sum of the corresponding observables for the free particle system and interaction terms. The most common example is the splitting of the Hamiltonian H for an interacting particle system as $H = H_0 + V$, where H_0 is the Hamiltonian for the same particle system without interactions and V,

☆ I dedicate this article to Professor William Wagner who first showed me the beauty and power of symmetry.

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the interaction term. The interaction term V is subjected to constraints arising from general requirements of quantum theory such as the self-adjointness of H, the existence and completeness of Møller wave operators and cluster decomposition. In addition to these, the overall symmetries that the interacting system is expected to obey also impose overarching restrictions on the perturbative terms added to observables. The primary goal of this paper to propose a general method for introducing interactions into a system of particles in a manner consistent with a given set of symmetry transformations. Further, we consider symmetries that can be encoded in finite dimensional, connected Lie groups (e.g., we exclude inversion symmetries). In the quantum mechanical setting, such symmetries are realized by a unitary, projective representation of the relevant Lie group in the Hilbert space of the system. Since the Hilbert space for nearly every realistic quantum system is separable and infinite dimensional (owing to the fact that it carries a unitary representation of a non-compact spacetime symmetry group), the subject of this study can be described as a perturbation theory for infinite dimensional unitary representations of finite dimensional connected Lie groups in Hilbert spaces. Our special focus will be the Galilei and Poincaré groups, the symmetry groups of non-relativistic and relativistic spacetimes, respectively.

If the interaction-free particle system obeys the symmetry structure of non-relativistic or relativistic spacetime, then the free Hamiltonian H_0 must belong to the Lie algebra of the Galilei or Poincaré group. As discussed in detail below, a basis for the Galilean algebra can be taken to consist of $\{H_0, \boldsymbol{P}_0, \boldsymbol{J}_0, \boldsymbol{K}_0, M_0\}$, where the momenta $\boldsymbol{P}_0$ and angular momenta $\boldsymbol{J}_0$ generate space translations and rotations respectively, the $\boldsymbol{K}_0$ generate Galilean boosts, and the mass M_0, a central extension, generates a one-parameter central subgroup. These operators must fulfill the characteristic commutation relations of the Galilean algebra. A basis for the Poincaré algebra can be taken to consist of $\{H_0, \boldsymbol{P}_0, \boldsymbol{J}_0, \boldsymbol{K}_0\}$, where, as in the Galilean case, $\boldsymbol{P}_0$ and $\boldsymbol{J}_0$ generate space translations and rotations, and $\boldsymbol{K}_0$ generate Lorentz boosts. These operators must satisfy the commutation relations of the Poincaré algebra. Now, if we require that the interacting system also satisfies the same spacetime symmetry as the interaction-free system, then the construction of the Hamiltonian as $H = H_0 + V$ must be done under the requirement that there exists a set of operators $\{\boldsymbol{P}, \boldsymbol{J}, \boldsymbol{K}, M\}$, possibly different from the free operators, which, along with H, satisfy the relevant commutation relations. This requirement clearly puts constraints on the interaction term V. It also tells us whether any of the other basis elements of the Lie algebra must be modified by interaction terms, along with the Hamiltonian. The structure of the Galilean algebra is such that an interaction term can be added to the Hamiltonian alone while keeping other generators interaction free. This is not the case for the Poincaré algebra. Along with H_0, at least some of the other generators such as $\boldsymbol{P}_0$ and $\boldsymbol{K}_0$ must change to accommodate interactions. Even for the non-relativistic case, the single perturbation $H = H_0 + V$ is not the only one allowed by the Galilean group. In particular, we will show below that momenta can also be altered by interactions in a manner consistent with Galilean symmetry.

The above discussion illustrates the general problem that we seek to address in this paper, namely constructing a procedure for introducing perturbations $\{\Delta A_i\}$ to a given set of operators $\{A_{0i}\}$ and defining new operators $A_i := A_{0i} + \Delta A_i$ such that both sets $\{A_{0i}\}$ and $\{A_i\}$ fulfill the same commutation relations. Since interactions in a system of particles yield such perturbative terms, we will often refer to perturbation terms ΔA_i as interactions. For the Poincaré algebra, the approach to this problem commonly taken within the setting of quantum field theories consists of starting with a Poincaré invariant Lagrangian density that is a function of fields and their derivatives and using the canonical formalism to construct interaction-incorporating generators. The interaction terms are constructed by means of local products of field operators (and their derivatives), a procedure that at times can be ambiguous and can lead to serious mathematical difficulties. In this study, we work at the more fundamental level of an operator Lie algebra and do not make the ancillary assumption of the existence of fields mediating interactions. Instead, we obtain the interaction-incorporating generators by means of introducing perturbations to the central elements of the Lie algebra.

In Section 2, we give the general construction for an arbitrary finite dimensional operator Lie algebra, followed by applications of the results to Galilean and Poincaré algebras in Section 3. In a followup paper [1], we use the results of this study to explicitly construct a Galilean algebra for an interacting two-particle system. One of the main goals of [1] is to show that interactions consistent with Galilean symmetry can be accommodated by way of mass operators with continuous spectra. Such operators