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Discrete accidental symmetry for a particle in a constant magnetic field on a torus

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ABSTRACT

A classical particle in a constant magnetic field undergoes cyclotron motion on a circular orbit. At the quantum level, the fact that all classical orbits are closed gives rise to degeneracies in the spectrum. It is well-known that the spectrum of a charged particle in a constant magnetic field consists of infinitely degenerate Landau levels. Just as for the $1/r$ and r^2 potentials, one thus expects some hidden accidental symmetry, in this case with infinite-dimensional representations. Indeed, the position of the center of the cyclotron circle plays the role of a Runge–Lenz vector. After identifying the corresponding accidental symmetry algebra, we re-analyze the system in a finite periodic volume. Interestingly, similar to the quantum mechanical breaking of CP invariance due to the θ-vacuum angle in non-Abelian gauge theories, quantum effects due to two self-adjoint extension parameters θ_x and θ_y explicitly break the continuous translation invariance of the classical theory. This reduces the symmetry to a discrete magnetic translation group and leads to finite degeneracy. Similar to a particle moving on a cone, a particle in a constant magnetic field shows a very peculiar realization of accidental symmetry in quantum mechanics.

1. Introduction

The fact that for some physical systems all bound classical orbits are closed leads to accidental degeneracies in the discrete energy spectrum of the corresponding quantum systems. Accidental symmetries are familiar from a particle moving in a $1/r$ or r^2 potential. In d spatial dimensions the system then has an $SO(d)$ rotational symmetry. In case of the $1/r$ potential, this symmetry is dynam-

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ically enhanced to an accidental $SO(d+1)$ symmetry, and for the r^2 harmonic oscillator potential it is enhanced to $SU(d)$. The accidental symmetries give rise to additional degeneracies in the discrete energy spectrum of the corresponding quantum systems, beyond the degeneracies one would expect based on rotation invariance alone [1,2]. The components of the Runge–Lenz vector [3] are the generators of the accidental symmetry algebras. The subject of accidental symmetry has been reviewed, for example, by McIntosh [4]. Recently, we have further investigated the phenomenon of accidental symmetries, by studying a particle confined to the surface of a cone and bound to its tip by a $1/r$ or r^2 potential [5]. When the deficit angle of the cone is a rational fraction of 2π, again all bound classical orbits are closed and there are accidental degeneracies in the energy spectrum of the quantum system. In this case the Runge–Lenz vector does not act as a self-adjoint operator in the domain of the Hamiltonian. Remarkably, as a consequence of this unusual property, the accidental $SU(2)$ symmetry has unusual multiplets with fractional (i.e. neither integer nor half-integer) spin.

An interesting example of an accidental symmetry involving a vector potential is cyclotron motion [6,7]. Also in this case, there is a deep connection between the fact that all bound classical orbits are closed and additional degeneracies in the discrete energy spectrum of the corresponding quantum system. As was already noted in [7], the center of the circular cyclotron orbit is a conserved quantity analogous to the Runge–Lenz vector in the Kepler problem. Also the radius of the cyclotron orbit is a conserved quantity directly related to the energy. Interestingly, while the two coordinates of the center are not simultaneously measurable, the radius of the circle has a sharp value in an energy eigenstate. In the cyclotron problem, translation invariance disguises itself as an "accidental" symmetry. As a consequence, the symmetry multiplets—i.e. the Landau levels—are infinitely degenerate. In order to further investigate the nature of the accidental symmetry, in [8] the charged particle in the magnetic field was coupled to the origin by an r^2 harmonic oscillator potential. This explicitly breaks translation invariance and thus reduces the degeneracy to a finite amount, while rotation invariance remains intact. In this paper, we do the opposite, i.e. we explicitly break rotation invariance, while leaving translation invariance (and hence the accidental symmetry) intact by putting the system on a torus. Interestingly, the Polyakov loops, which are a consequence of the non-trivial holonomies of the torus, give rise to non-trivial Aharonov–Bohm phases which are observable at the quantum but not at the classical level. Analogous to the quantum mechanical breaking of CP invariance due to the θ-vacuum angle in non-Abelian gauge theories, here two self-adjoint extension parameters θ_x and θ_y explicitly break the continuous translation invariance of the classical problem down to a discrete magnetic translation group [9]. This reduces the degeneracy to a finite amount, and allows us to further investigate the nature of the accidental symmetry. In particular, just like for motion on a cone [5], symmetry manifests itself in a rather unusual way in this quantum system. In particular, due to its relevance to the quantum Hall effect, the Landau level problem has been studied very extensively (for a recent review see [10]). For example, the problem has already been investigated on a torus in [11,12], however, without emphasizing the accidental symmetry aspects. In this paper, we concentrate entirely on those aspects, thus addressing an old and rather well-studied problem from an unconventional point of view.

The rest of the paper is organized as follows. In Section 2 the cyclotron problem is reviewed in the infinite volume, with special emphasis on its oscillator algebras and accidental symmetry generators. In Section 3 the system is put on a torus and the unusual manifestation of the accidental symmetry is worked out. Section 4 contains our conclusions.

2. Particle in the infinite volume

In this section we review the standard knowledge about a non-relativistic particle moving in a constant magnetic field in the infinite volume. We proceed from a classical to a semi-classical, and finally to a fully quantum mechanical treatment. In particular, we emphasize the symmetry aspects of the problem with a focus on accidental symmetries. This section is a preparation for the case of a finite periodic volume to be discussed in the next section. In the following, we will use natural units in which $\hbar = c = 1$.